Sharper Alpha

New testable implications for asset pricing

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Abstract

Traditional alpha-based tests face substantial noise when estimated on single stocks. Using diversified test portfolios helps to reduce this noise but incurs the costs of inducing aggregation error and reducing cross-sectional variability. To address these shortcomings, we propose a more efficient statistic, a *sharper alpha*, that reduces estimation noise for single stocks. We find that, while sharper alphas estimated on portfolios are similar to traditional OLS alphas, they provide significant noise reduction at the stock-level and reveal cross-sectional patterns that were not visible before.

Keywords: Asset Pricing, Model Evaluation, Estimating Alpha, APT, Efficiency Loss

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Patterns and anomalies in the cross-section of expected returns have inspired hundreds of pricing factors (Harvey and Y. Liu 2019). Collectively referred to as the factor zoo (Cochrane 2011), each "inhabitant" proposes its own competing theory for the economic drivers of alpha – the part of risk premia beyond what the CAPM can explain. These competing models relate cross-sectional patters in alpha to omitted risk (Constantinides and Duffie 1996; Merton 1973), financial frictions or constraints (Asness et al. 2019; Frazzini and Pedersen 2014; Jianan Liu, Stambaugh, and Yuan 2016; Stambaugh, Yu, and Yuan 2015), investor preferences (Bansal and Yaron 2004), beliefs (Andrei, Cujean, and Fournier 2019), and behavioral biases (Bali et al. 2017; Barberis and Thaler 2003; Hirshleifer 2015).

Evaluating these competing models is difficult due to the estimation noise in alphas. Indeed, the high amount of noise in single stock alphas makes it hard to detect any cross-sectional patterns. The traditional solution is to form characteristic-sorted portfolios, where each portfolio diversifies away idiosyncratic noise to produce more precise estimates of alpha (Blume 1970). However, aggregating into portfolios also limits cross-sectional variability and induces aggregation error in the form of strongly correlated factor loadings across the cross-section of portfolio returns (Lewellen, Nagel, and J. Shanken 2010). As a result, competing economic models become hard to distinguish in portfolio-based tests, as they seem to all perform equally well (Daniel and Titman 2012). Effectively, portfolios help average away the estimation noise, but they also average away the very details on the cross-section of risk premia that we wish to study (Ang, Jun Liu, and Schwarz 2020).

To address these shortcomings, this paper develops a new test statistic, *sharper alpha*, that reduces estimation noise without resorting to forming portfolios. We show that sharper alphas provide significant improvements over OLS alphas for both portfolios and single stocks. At the stock level in particular, OLS alphas do not reveal statistically significant relations to characteristics such as size, value, profitability, investment, and betting-against-beta. The sharper alphas do. Our approach therefore detects statistically significant alpha without sacrificing stock-level variation in characteristics.

The main insight behind our approach is that, since alpha is the difference between expected return and the required return under some benchmark model, we can improve estimates of alpha by using better estimates for the expected return. Assuming that we observe the returns on a Sharpe ratio maximizing tangency portfolio, we can obtain a better estimate of a stock's expected return by projecting its return onto the return of the tangency portfolio. This estimate is more precise than a simple sample average because any unpriced idiosyncratic noise in the stock return is uncorrelated with the tangency portfolio, and is therefore filtered out by the projection.

Using the tangency portfolio as a "better yard stick" for measuring risk premia, we propose our sharper alpha statistic and provide its asymptotic and finite sample distributions. We show that it is a consistent estimator like OLS alpha, but is asymptotically more efficient. Our approach can be applied to sharpen not only CAPM alphas, but also alphas from multi-factor models. In addition, we develop a sharper multi-asset joint test (as in Gibbons, Ross, and J. Shanken 1989).

Of course we do not actually observe the tangency portfolio. Since the tangency returns need to be estimated, we need to verify that the sharper alphas are robust against potential measurement error. To address this concern, we provide sufficient conditions under which measurement error in the tangency returns causes only an attenuation bias that shrinks all sharper alphas towards zero. Under these conditions, our sharper alphas provides a lower bound on the magnitude of the true alphas. Therefore, we do not need perfect tangency returns to produce meaningful sharper alphas that can identify cross-sectional patterns in the true alphas.

To apply our new statistic, we first construct a U.S. proxy for the tangency portfolio from 1967 to 2019. We build on Andrei, Cujean, and Fournier (2019) and use a 30 year rolling window to build the tangency portfolio from daily returns of portfolios sorted on size, value, profitability, investment, and momentum. The estimated portfolio weights are stable over time, generating an average daily turnover of around 1.24% (corresponding to an average holding period of 80 trading days). Even though the tangency portfolio is estimated only with ex-ante available information, it achieves a high Sharpe ratio and earns the market risk premium at less than a third the market's volatility. Further, we verify that it satisfies the conditions to produce sharper alphas that are robust against measurement error.

Second, we validate our sharper alphas against traditional OLS alphas at the portfolio level, where much of the idiosyncratic noise is already diversified away. Examining daily returns from 1967 to 2019, we find that sharper alphas track the OLS alphas across a range of characteristic sorted portfolios. Further, we observe improved statistical significance across the board. Thus, at the portfolio level where we have more reliable OLS alphas, we confirm that the sharper alphas improve efficiency without introducing any bias.

Third, we estimate our sharper alphas at the stock level and show that they can better detect true alphas. At the stock level, less than 0.12% of OLS alphas are statistically different from zero. In contrast, over 27.88% of sharper alphas are statistically significant (with t-statistics greater than 3). Sharper alphas thus provide statistically significant results without sacrificing cross-sectional variability by pooling returns into diversified portfolios.

Fourth, we find that sharper alphas reveal new details in these cross-sectional patterns, details that are lost to estimation noise in traditional OLS alphas. For the profitability anomaly in particular, we find that a large part of the positive profitability-alpha relation is driven by firms with negative profitability (as measured by ROE). This new empirical insight provides useful guidance for finance theory. It suggests that theory should examine mechanisms that are particularly important to these firms that are still in operation despite being unprofitable. Examples include models that study growth options (Berk, Green, and Naik 1999) and how the value of these growth options interact with business cycles (Zhang 2005) or with variance risk premia (Ericsson, Jo, and Lotfaliei 2020).

Our work is part of a growing empirical asset pricing literature that advocates the use of large cross-sections of test assets. Lewellen, Nagel, and J. Shanken (2010) show that portfolio grouping generates aggregation error and offers mitigation measures. Chordia, Goyal, and

J. A. Shanken 2019 develop errors-in-variables corrections for cross-sectional regressions of individual stock returns. Gagliardini, Ossola, and Scaillet (2016) and Chaieb, Langlois, and Scaillet (2020) estimate time-varying risk premia from large unbalanced panels of individual stocks. Ang, Jun Liu, and Schwarz (2020) argues that the loss in cross-sectional variability from using portfolios instead of individual stocks results in an overall loss in efficiency for cross-sectional tests. Barras (2019) proposes a large-scale approach that strikes a balances between limiting aggregation error in portfolios and reducing noise in individual stocks by forming a large number of micro-portfolios.

Our paper complements this literature because our sharper alpha can be used to sharpen any existing, alpha-based tests, whether they are done at the portfolio, micro-portfolio, of stock level. Our sharper alpha improves statistical power both for portfolios and individual stocks. Using sharper alphas as inputs could therefore contribute towards improving the statistical efficiency of a wide range of asset pricing tests.

Our paper is also related to work on risk factor and beta decompositions. Campbell and Shiller 1988 and Binsbergen and Koijen 2010 provide a framework to decompose market risk into a cash flow component and a discount rate component. Campbell and Vuolteenaho (2004) apply this decomposition to market beta and argue that stocks with high cash flow beta generate alpha because cash flow beta earns a higher risk premium than discount rate beta. Our paper provides an alternative decomposition of market beta into i) a tangency factor, which earns the maximum risk premium per unit of risk, and ii) an unpriced factor, which earns no risk premium. Using this decomposition, we construct a stock's sharper alpha from the difference in both components of market beta, and show that this difference is less exposed to estimation noise than the traditional OLS alpha.

The organization of the paper is as follows. Section 1 develops the sharper alpha statistic assuming that the tangency portfolio is perfectly observed. Section 2 estimates the tangency portfolio and show that sharper alphas are robust against potential measurement error. Section 3 demonstrates efficiency gains by comparing sharper alphas to traditional OLS alphas on both single-stocks and characteristic-sorted portfolios. Section 4 concludes. The appendix provides proofs and additional empirical tests.

1 Model

1.1 Setup

We consider a market with N stocks indexed by $n \in \{1, ..., N\}$. We denote by

$$\underset{N\times 1}{\boldsymbol{r}} \equiv [r_1, ..., r_N]^\top \tag{1}$$

the $N \times 1$ vector of stock excess returns. We make the following assumptions.

Assumption 1. *Excess stock returns are distributed with finite mean* $\underset{N\times 1}{\boldsymbol{\mu}} \equiv \mathrm{E}[\boldsymbol{r}] = [\mu_1, ..., \mu_N]^\top$ *and covariance matrix* $\underset{N\times N}{\Sigma} \equiv \mathrm{VAR}[\boldsymbol{r}]$.

Assumption 2. *The covariance matrix Σ is invertible: there are no redundant securities.*

Let $\boldsymbol{w}_{\mathsf{M}}$ be the vector of market portfolio weights and

$$r_{\mathsf{M}} \equiv \boldsymbol{w}_{\mathsf{M}}^\top \boldsymbol{r} \tag{2}$$

its excess returns. We denote the market portfolio's risk premium and variance by

$$\mu_{\mathsf{M}} \equiv \mathrm{E}[r_{\mathsf{M}}] = \boldsymbol{w}_{\mathsf{M}}^\top \boldsymbol{\mu} \tag{3}$$

$$\sigma_{\mathsf{M}}^2 \equiv \mathrm{VAR}[r_{\mathsf{M}}] = \boldsymbol{w}_{\mathsf{M}}^\top \Sigma \boldsymbol{w}_{\mathsf{M}}. \tag{4}$$

Further, let $\boldsymbol{w}_\tau$ be the vector of tangency portfolio weights that maximizes Sharpe Ratio while still earning the same risk premia as the market. Using the portfolio rule from Markowitz 1952, the tangency portfolio is constructed as

$$\boldsymbol{w}_\tau = \arg\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^\top \Sigma \boldsymbol{w}}} \text{ s.t. } \mathrm{E}[r_\tau] = \mu_{\mathsf{M}} \tag{5}$$

$$= \frac{\Sigma^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}} \cdot \mu_{\mathsf{M}} \tag{6}$$

and generates excess returns

$$r_\tau \equiv \boldsymbol{w}_\tau^\intercal \boldsymbol{r}. \tag{7}$$

The tangency portfolio's risk premium and variance are then

$$\mu_\tau \equiv \mathrm{E}\left[r_\tau\right] = \mu_{\mathsf{M}} \tag{8}$$

$$\sigma_\tau^2 \equiv \mathrm{VAR}\left[r_\tau\right] = \frac{\mu_{\mathsf{M}}^2}{\boldsymbol{\mu}^\intercal \Sigma^{-1} \boldsymbol{\mu}} \tag{9}$$

$$\sigma_{\tau,\mathsf{M}} \equiv \mathrm{COV}\left[r_{\mathsf{M}}, r_\tau\right] = \frac{\mu_{\mathsf{M}}^2}{\boldsymbol{\mu}^\intercal \Sigma^{-1} \boldsymbol{\mu}}. \tag{10}$$

Effectively, the tangency portfolio simply rescales the Markowitz, maximum Sharpe ratio portfolio to earn market risk premium at the lowest volatility possible.

We can then decompose the market portfolio into the tangency portfolio and an orthogonal component that earns zero risk premia. We call this component the unpriced factor. We define its portfolio weights as

$$w_{\mathsf{U}} \equiv w_{\mathsf{M}} - w_\tau, \tag{11}$$

which earns $w_{\mathsf{U}}^\intercal \boldsymbol{\mu} = 0$ by construction. The unpriced factor has returns $r_{\mathsf{U}} \equiv w_{\mathsf{U}}^\intercal \boldsymbol{r}$ and variance $\sigma_{\mathsf{U}}^2 \equiv w_{\mathsf{U}}^\intercal \Sigma w_{\mathsf{U}}$.

Since the market index is the sum of its orthogonal components

$$r_{\mathsf{M}} = r_\tau + r_{\mathsf{U}}, \tag{12}$$

the market variance is can be expressed as the sum of the component variances

$$\sigma_{\mathsf{M}}^2 = \sigma_{\tau}^2 + \sigma_{\mathsf{U}}^2. \tag{13}$$

The proportion of market variance that is priced, $\frac{\sigma_{\tau}^2}{\sigma_{\mathsf{M}}^2}$, is equal to one when the CAPM holds. As CAPM fails, however, the market takes on some proportion of unpriced variance $\frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2} > 0$.

1.2 Return Decomposition

To relate alphas to the tangency factor, we express stock returns in terms of their loadings on the tangency and unpriced factors. We define a stock's tangency beta as

$$\beta_{\tau,n} \equiv \frac{\mathrm{cov}\left[r_{\tau}, r_n\right]}{\sigma_{\tau}^2}, \tag{14}$$

which measures risk loading on the market's priced component r_{τ}.

Unpriced beta is defined as

$$\beta_{\mathsf{U},n} \equiv \frac{\mathrm{cov}\left[r_{\mathsf{U}}, r_n\right]}{\sigma_{\mathsf{U}}^2} \tag{15}$$

which measures risk coming from the unpriced component r_{U}.

We can define the residual that is uncorrelated with both components of the market as

$$\varepsilon_n \equiv r_n - \beta_{\tau,n} r_{\tau} - \beta_{\mathsf{U},n} r_{\mathsf{U}}, \tag{16}$$

and rewrite stock returns as

$$r_n = \beta_{\tau,n} r_{\tau} + \beta_{\mathsf{U},n} r_{\mathsf{U}} + \varepsilon_n. \tag{17}$$

The stock's risk premium is depends only on its tangency beta

$$\mu_n = \beta_{\tau,n} \cdot \mu_{\mathsf{M}}. \tag{18}$$

Our decomposition thus breaks stock returns into three orthogonal components: a tangency component generating both covariance with the market and risk premia, an unpriced component generating covariance risk but no risk premia, and the residual component generating neither. This decomposition always holds.

1.3 Components of Beta and Alpha

We now apply the return decomposition to betas and alphas in order to derive the sharper alpha statistic. We start with the simple case where alphas are measured with respect to the CAPM. We then show that the results generalize to multi-factor models.

Consider the CAPM regression

$$r_n = \alpha_n + \beta_n r_{\mathsf{M}} + u_n \tag{19}$$

and define the CAPM beta and alpha as

$$\beta_n \equiv \frac{\mathrm{cov}\left[r_n, r_{\mathsf{M}}\right]}{\sigma^2_{\mathsf{M}}} \tag{20}$$

$$\alpha_n \equiv \mathrm{E}\left[r_n\right] - \beta_n \mu_{\mathsf{M}}. \tag{21}$$

Beta Decomposition Combining (12), (17) and (20), we decompose the CAPM beta in terms of covariance with each component:

$$\beta_n = \frac{\mathrm{cov}\left[\beta_{\tau,n} r_\tau + \beta_{\mathsf{U},n} r_{\mathsf{U}} + \varepsilon_n, r_\tau + r_{\mathsf{U}}\right]}{\sigma^2_{\mathsf{M}}} \tag{22}$$

Proposition 1. *The CAPM beta of stock-n reflects a weighted average of the component betas*

$$\beta_n = \left(1 - \frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2}\right) \beta_{\tau,n} + \frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2} \beta_{\mathsf{U},n}. \tag{23}$$

Intuitively, betas tend to reflect the priced, tangency betas when $\frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2}$ is close to 0 and the market portfolio consists mostly of priced risk. However, when the CAPM fails and the market portfolio has a higher proportion of unpriced risk, market betas start to pick up unpriced betas instead.

This decomposition of market beta is key to constructing a new measure of alpha. By definition, alpha is the difference between the stock's risk premium, which according to (18) is proportional to the stock's tangency beta, and the stock's required risk premium, which is proportional to its market beta. Given (23), we can therefore express alpha as a function of the stock's tangency and unpriced betas.

Alpha The CAPM treats both components of beta as priced risk, while only tangency betas earn risk premia. Using (21) and (17), we can calculate alpha from this wedge between the priced and unpriced betas.

Proposition 2. *The CAPM alpha reflects the difference between the priced and unpriced components of beta*

$$\alpha_n = \mathrm{E}\left[\beta_{\tau,n} r_\tau + \beta_{\mathsf{U},n} r_{\mathsf{U}} + \varepsilon_n\right] - \beta_n \mu_{\mathsf{M}} = \left(\beta_{\tau,m} - \beta_n\right) \mu_{\mathsf{M}} \tag{24}$$

$$= \frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2} \left(\beta_{\tau,n} - \beta_{\mathsf{U},n}\right) \mu_{\mathsf{M}} \tag{25}$$

Stocks with disproportionately high unpriced betas appear to require higher expected returns than actually earned, thus resulting in negative alphas. On the other hand, stocks with disproportionately high tangency betas appear to earn higher premia than required by the CAPM, and get positive alphas instead.

We note that alphas are scaled by the proportion of unpriced variance $\frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2}$. When the CAPM holds and $\frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2}$ is equal to zero, so all alphas to shrink zero as well.

1.4 Statistical Inference

Using our beta decomposition (23), we construct our sharper alpha statistic and show that it is asymptotically more efficient that the traditional OLS alpha. We then generalize our sharper alpha to multi-factor models, and to joint tests with multiple assets.

1.4.1 Sharper Alpha with the CAPM

For a finite sample of size T, denote by $r_{n,t}$ the observed excess return on asset n for time $t = 1, ..., T$ and $r_{\mathsf{M},t}$ is the excess market return.

Under the CAPM, we write the return as

$$r_{n,t} = \alpha_n + \beta_n \cdot r_{\mathsf{M},t} + u_{n,t} \tag{26}$$

where α_n, β_n, and $u_{n,t}$ are the CAPM alpha, beta, and residuals.

To construct the *sharper alpha*, $\hat{\alpha}_p^*$, we estimate the tangency beta and the unpriced beta from the joint regression

$$r_{n,t} = \beta_{\tau,n} \cdot r_{\tau,t} + \beta_{\mathsf{U},n} \cdot r_{\mathsf{U},t} + \varepsilon_{n,t} \tag{27}$$

and apply the beta decomposition from (23) to estimate alpha as

$$\hat{\alpha}_n^* = \frac{\hat{\sigma}_{\mathsf{U}}^2}{\hat{\sigma}_{\mathsf{M}}^2} \bar{r}_{\mathsf{M}} \left(\hat{\beta}_{\tau,p} - \hat{\beta}_{\mathsf{U},p} \right) . \tag{28}$$

Now that we have our sharper alpha estimator, we derive its asymptotics to show that it is consistent and efficient.

Proposition 3. *As $T \to \infty$, the asymptotic variance of the sharper alpha $\hat{\alpha}_p^*$ is given by*

$$\text{VAR}\left[\hat{\alpha}_n^*\right] = \left(\frac{\sigma_{\mathsf{U}}^2}{\sigma_{\mathsf{M}}^2}\mu_{\mathsf{M}}\right)^2 w^\intercal V^* w \tag{29}$$

where $w = [1, -1]^\intercal$ and V^* is the asymptotic covariance matrix of the tangency and unpriced betas. Using some consistent estimator $\hat{V}^*$, we compute the t-statistic as

$$\hat{t}_n^* = \frac{\hat{\alpha}_n^*}{\frac{\hat{\sigma}_{\mathsf{U}}^2}{\hat{\sigma}_{\mathsf{M}}^2}\bar{r}_{\mathsf{M}}\sqrt{w^\intercal \hat{V}^* w}} = \frac{\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}}{\sqrt{w^\intercal \hat{V}^* w}} \tag{30}$$

which is asymptotically normal

$$\hat{t}_n \xrightarrow{d} N\left(\frac{\alpha_n}{\sqrt{\text{VAR}\left[\hat{\alpha}_n^*\right]}}, 1\right). \tag{31}$$

Intuitively, the t-statistic on sharper alpha is simply a test on the difference between the tangency beta and the unpriced beta. Since true alphas are proportional to this difference in betas, our sharper alpha is indeed a consistent estimator of the true alpha. Having shown consistency, we now check whether the sharper alphas are in fact more efficient than the OLS alphas.

Proposition 4. *With serially uncorrelated and homoscedastic error terms $\varepsilon_{n,t}$ where $\text{COV}\left[\varepsilon_{n,t}, \varepsilon_{n,t-j}\right] = 0 \;\forall\; j \neq 0$ and $\text{VAR}\left[\varepsilon_{n,t}\right] = \sigma_\varepsilon^2 \;\forall\; t$, the sharper alpha $\hat{\alpha}_p^*$ is more efficient than the OLS estimate $\hat{\alpha}_p$. Their asymptotic variance ratio is*

$$\frac{\text{VAR}\left[\hat{\alpha}_n\right]}{\text{VAR}\left[\hat{\alpha}_n^*\right]} = \left(\left(\frac{\alpha_n^2}{\sigma_\varepsilon^2}\frac{\sigma_{\mathsf{M}}^2}{\mu_{\mathsf{M}}^2} + 1\right) \cdot \frac{\sigma_\tau^2}{\sigma_{\mathsf{U}}^2} + 1\right) \cdot \left(\frac{\sigma_\tau^2}{\mu_{\mathsf{M}}^2} + 1\right) > 1 \tag{32}$$

This is the key result of our paper: we show that sharper alphas are in fact sharper than OLS alphas. By first projecting returns on the tangency and unpriced factors, we obtain a more efficient estimator because the tangency and unpriced betas are less affected by idiosyncratic noise in the stock returns. These asymptotic efficiency gains are higher when the market has less priced variance σ_τ^2 relative to unpriced variance σ_{U}^2, and when the magnitude of alpha α_n^2 is large relative to the residual noise σ_ε^2. These results suggest that the sharper alpha $\hat{\alpha}_n^*$ provides more statistical power in assessing asset pricing models precisely where it is most needed.

1.4.2 Sharper Alpha with multi-factor models

We now apply our results to a multi-factor model

$$r_{n,t} = \alpha_n + \beta_n \cdot r_{\mathsf{M},t} + \sum_{k=1}^{K} \beta_{k,n} \cdot f_{k,t} + u_{n,t} \tag{33}$$

where $f_{k,t}$ is the excess return on some traded factor or factor mimicking portfolio.

We start by reducing (33) to a simple regression between the parts of asset and market returns not spanned by the additional factors. Define as $\tilde{r}_{n,t}$ and $\tilde{r}_{\mathsf{M},t}$ the residuals from regressing $r_{n,t}$ and $r_{\mathsf{M},t}$ on the additional factors $f_{1,t}, ..., f_{K,t}$. By the Frisch-Waugh-Lovell Theorem (Frisch and Waugh 1933; Lovell 1963), estimating the simple regression

$$\tilde{r}_{n,t} = \alpha_n + \beta_n \cdot \tilde{r}_{\mathsf{M},t} + u_{n,t} \tag{34}$$

yields coefficients and residuals that are numerically identical to estimating regression (33).

By the same logic, we can simplify the joint regression

$$r_{n,t} = \beta_{\tau,n} \cdot r_{\tau,t} + \beta_{\mathsf{U},n} \cdot r_{\mathsf{U},t} + \sum_{k=1}^{K} \beta_{k,n} \cdot f_{k,t} + \varepsilon_{n,t} \tag{35}$$

to

$$\tilde{r}_{n,t} = \beta_{\tau,n} \cdot \tilde{r}_{\tau,t} + \beta_{\mathsf{U},n} \cdot \tilde{r}_{\mathsf{U},t} + \varepsilon_{n,t} \tag{36}$$

where $\tilde{r}_{\tau,t}$ and $\tilde{r}_{\mathsf{U},t}$ are the residuals from regressing the tangency and unpriced factors on the additional factors $f_{1,t}, \ldots, f_{K,t}$.

Using the modified CAPM regressions (34) with (36), we construct the multi-factor sharper alpha

$$\hat{\alpha}_n^* = \frac{\hat{\sigma}_{\mathsf{U}}^2}{\hat{\sigma}_{\mathsf{M}}^2} \bar{r}_{\mathsf{M}} \left(\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n} \right) \tag{37}$$

where $\hat{\sigma}_{\mathsf{U}}^2$ and $\hat{\sigma}_{\mathsf{M}}^2$ are the sample variances of $\tilde{r}_{\mathsf{U},t}$ and $\tilde{r}_{\mathsf{M},t}$, and $\bar{r}_{\mathsf{M}}$ is the sample mean of $\tilde{r}_{\mathsf{M},t}$.

Both our consistency and efficiency results from the CAPM case apply to this multi-factor version of sharper alpha.

Proposition 5. *With $\hat{\beta}_{\tau,n}$ and $\hat{\beta}_{\mathsf{U},n}$ estimated from the joint regression* (35)*, the sharper alpha has a t-statistic of*

$$\hat{t}_p = \frac{\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}}{\sqrt{w^{\mathsf{T}} \hat{V}^* w}} \tag{38}$$

where $w = [1, -1]^{\mathsf{T}}$ and $\hat{V}^*$ is some consistent estimator for the covariance matrix of the betas.

Intuitively, we can estimate the multi-factor alpha by applying CAPM results to the part of returns not spanned by the K additional factors. The efficiency gains from sharper alphas can therefore be applied to any linear factor model.

1.4.3 Joint Tests of No Mispricing

In addition to sharpening alphas on individual stocks, we can use sharper alphas as inputs to improve existing alpha-based tests. We take the multi-asset GRS test and derive a "sharpened" GRS test based on our sharper alphas.

Consider a panel of N assets, with the N-vectors of tangency betas $\hat{\boldsymbol{\beta}}_\tau = \left[\hat{\beta}_{\tau,1}, \ldots, \hat{\beta}_{\tau,N}\right]^{\mathsf{T}}$ and unpriced betas $\hat{\boldsymbol{\beta}}_{\mathsf{U}} = \left[\hat{\beta}_{\mathsf{U},1}, \ldots, \hat{\beta}_{\mathsf{U},N}\right]^{\mathsf{T}}$.

Proposition 6. *The joint test statistic with K factors and N test assets is*

$$J_{\mathrm{S}} = \left[\hat{\boldsymbol{\beta}}_{\tau} - \hat{\boldsymbol{\beta}}_{\mathsf{U}}\right]^{\intercal} \left[\mathrm{VAR}\left[\hat{\boldsymbol{\beta}}_{\tau} - \hat{\boldsymbol{\beta}}_{\mathsf{U}}\right]\right]^{-1} \left[\hat{\boldsymbol{\beta}}_{\tau} - \hat{\boldsymbol{\beta}}_{\mathsf{U}}\right]. \tag{39}$$

Assuming the factors are exogenous and that the error term have no serial correlation,

$$J_{\mathrm{S}} = T \cdot \left(1 + \frac{\bar{r}_{\mathsf{M}}^2}{\hat{\sigma}_{\mathsf{M}}^2}\right)^{-1} \cdot \left(1 + \frac{\bar{r}_{\tau}^2}{\hat{\sigma}_{\tau}^2}\right) \cdot \frac{\hat{\sigma}_{\tau}^2 \hat{\sigma}_{\mathsf{U}}^2}{\hat{\sigma}_{\mathsf{M}}^2} \cdot \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \quad \hat{\boldsymbol{\beta}}_{\mathsf{U}}\ \right]^{\intercal} \mathrm{VAR}\left[\hat{u}\right]^{-1} \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \quad \hat{\boldsymbol{\beta}}_{\mathsf{U}}\ \right] \tag{40}$$

is asymptotically distributed χ_N^2 under the null hypothesis of no mispricing. Further assuming that the errors are joint normal, the finite sample analogue is

$$F_{\mathrm{S}} = \frac{T - N - K}{N} \frac{J_{\mathrm{S}}}{T - K - 1} \overset{d}{\sim} F_{N, T-N-K}. \tag{41}$$

With this sharper GRS test, we can test for alphas against one or many factors, on either one or many assets jointly.

Overall, our approach offers efficiency gains over traditional OLS alphas and can be just as easily applied to a wide range of models using any existing alpha-based test.

2 The tangency portfolio and measurement error

In this section, we implement our decomposition of market returns using rolling estimates of the tangency factor. We then investigate how measurement error in the tangency factor affects the sharper alphas. We focus our empirical analysis on daily U.S. stock returns from CRSP, with accounting variables from COMPUSTAT, between 1967 and 2019.

2.1 Estimating the tangency portfolio

To estimate the tangency portfolio weights, we first construct the tangency portfolio using characteristic sorted portfolios, similar to Andrei, Cujean, and Fournier 2019. We use 25 value-size sorted portfolio as in Fama and French 1992, 18 profitability and investment sorted portfolios as in Hou, Xue, and Zhang 2017, and 10 portfolios sorted on momentum. For each day t from 1967 to 2019, we use a 30-year rolling window of daily data to compute the sample mean $\hat{\boldsymbol{\mu}}_t$ and covariance matrix $\hat{\Sigma}_t$ on the excess returns of these characteristic-sorted portfolios. We compute the average market excess return $\hat{\mu}_{\mathsf{M},t}$ over the same sample period, and estimate the tangency portfolio weights as

$$\hat{\boldsymbol{w}}_{\tau,t} = \hat{\Sigma}_t^{-1} \hat{\boldsymbol{\mu}}_t \cdot \frac{\hat{\mu}_{\mathsf{M},t}}{\hat{\boldsymbol{\mu}}_t^{\mathsf{T}} \hat{\Sigma}_t \hat{\boldsymbol{\mu}}_t}. \tag{42}$$

Given the market portfolio weights $\boldsymbol{w}_{\mathsf{M},t}$, we back out the unpriced residual as

$$\hat{\boldsymbol{w}}_{\mathsf{U},t} = \boldsymbol{w}_{\mathsf{M},t} - \hat{\boldsymbol{w}}_{\tau,t}. \tag{43}$$

Finally, using the estimated tangency and unpriced portfolio weights, we decompose the time $t+1$ market return as

$$r_{\mathsf{M},t+1} = \hat{\boldsymbol{w}}_{\tau,t}\boldsymbol{r}_{t+1} + \hat{\boldsymbol{w}}_{\mathsf{U},t}\boldsymbol{r}_{t+1} \tag{44}$$

$$= \hat{\boldsymbol{r}}_{\tau,t+1} + \hat{\boldsymbol{r}}_{\mathsf{U},t+1}. \tag{45}$$

To gauge the out of sample validity of our decomposition of market returns, we compare the performance of our ex-ante estimated portfolios to what what an ex-post optimal portfolio can achieve with perfect hindsight. Figure 1 plots a snapshot of the annualized risk premia and volatilities for a sample of characteristic sorted portfolios, along with the in-sample mean-variance frontier. The green and red dots show the ex-ante, estimated tangency and unpriced factors. The green and red targets show the ex-post realized tangency factor and the corresponding unpriced factor. Perhaps surprisingly, even though the tangency portfolio weights are restricted to using ex-ante available information, the tangency portfolio did not miss the target by much.

If we have no measurement error, we should in theory expect to see tangency returns earning the same risk premia as the market at a lower volatility. We should also expect the residual, unpriced factor to earn zero risk premium and to be uncorrelated to the tangency factor. We find that this is exactly how our estimated factors behave empirically.

Figure 2 plots the cumulative log returns of the tangency and unpriced portfolios. The tangency factor generates the market risk premium at less than a third of the market's volatility, while the unpriced factor earns zero risk premium. We test this more formally in Table 1, where we show that the tangency factor earns a significantly higher Sharpe ratio than the market and that the unpriced factor does not earn a significant premium.

Moreover, in Table 2, we compare the market to its tangency and unpriced components. We see that the estimated tangency factor only accounts for less than 7% of the market variance. A surprisingly large portion of market variance therefore seems to be unpriced.

Finally, in Table (3), we estimate the correlation matrix between the market portfolio and its components. We find that the estimated unpriced factor is uncorrelated with the tangency factor, which is consistent with having low measurement error

Given the long, 30 year rolling window used, the tangency portfolio weights move slowly over time. The estimated tangency portfolio has a average turnover of 1.24%, which corresponds to an average holding period of 80 trading days or approximately 4 months. The ex-ante tangency portfolio's ability to keep earning a high Sharpe ratio out of sample shows that the two components of the market portfolio are remarkably stable over time. For whatever shortcoming the CAPM may have, it consistently fails in a stable and persistent way.

2.2 Sharper alpha under measurement error

Now that we have an empirical candidate for the tangency portfolio, it is important to check whether sharper alphas are robust against measurement error in the tangency returns.

Denote by ξ the measurement error in the tangency return. The two estimated components of the market are

$$\hat{r}_\tau = r_\tau + \xi \tag{46}$$

$$\hat{r}_{\mathsf{U}} = r_{\mathsf{M}} - \hat{r}_\tau = r_{\mathsf{U}} - \xi. \tag{47}$$

The presence of this measurement error can generate an errors-in-variables bias in the estimated betas. To pin down this bias in closed-form, it helps to restrict the measurement error to be consistent with our empirical observations. Specifically, we make the two following restrictions, both of which are empirically supported by our estimated factor returns. First, we require that the estimated unpriced factor earns zero risk premium

$$\mathrm{E}\left[\hat{r}_{\mathsf{U}}\right] = 0 \tag{48}$$

which is consistent with the tangency portfolio earning the same risk premium as the market, as we see in Table 1. Second, we assume that the estimated factors are uncorrelated with each other

$$\operatorname{cov}\left[\hat{r}_{\tau}, \hat{r}_{\mathsf{U}}\right] = 0 \tag{49}$$

which is consistent with Table (3). Given these assumptions, we now provide a close-formed expression for the bias from measurement error in the tangency returns.

Proposition 7. *If* $\mathrm{E}\left[\hat{r}_{\tau}\right] = \mathrm{E}\left[r_{\tau}\right]$ *and* $\operatorname{cov}\left[\hat{r}_{\tau}, \hat{r}_{\mathsf{U}}\right] = 0$*, then the sharper alpha* $\hat{\alpha}^*$ *converges towards*

$$\hat{\alpha}_n^* \xrightarrow[T \to \infty]{} (1-\pi_1)(1-\pi_2) \cdot \alpha_n + \pi_1 \cdot \beta_{\xi,n} \cdot \mu_{\mathsf{M}} \tag{50}$$

where

$$\pi_1 = \frac{\mathrm{E}\left[\xi^2\right]}{\mathrm{E}\left[\hat{r}_{\tau}^2\right]} = \frac{\sigma_{\xi}^2}{\mu_{\mathsf{M}}^2 + \sigma_{\tau}^2 + \sigma_{\xi}^2} \in [0,1] \tag{51}$$

is the amount of noise in the estimated tangency factor,

$$\pi_2 = \frac{\mathrm{E}\left[\xi^2\right]}{\mathrm{E}\left[r_{\mathsf{U}}^2\right]} = \frac{\sigma_{\xi}^2}{\sigma_{\mathsf{U}}^2} \in [0,1]$$

is the amount of noise relative to the unpriced factor, and $\beta_{\xi,n} = \frac{\operatorname{cov}[\xi, \varepsilon_n]}{\sigma_{\xi}^2}$ *is each stock's direct loading on the measurement error.*

Proposition (7) breaks down the errors-in-variables bias into two components. The first component is an attenuation bias where more measurement error (σ_{ξ}^2) increases the noise ratios (π_1,π_2) and shrinks sharper alpha towards zero. Because the attenuation bias affects all sharper alphas proportionally, the attenuated estimates provide lower bounds on the cross-sectional differences in true alphas. Figure (A.1) in the appendix illustrates how this attenuation varies with the amount of measurement error σ_{ξ}.

The second component is an omitted variable bias that could pose a more significant problem. Stock residuals that correlate with the measurement error incorrectly show up in sharper alphas. The more noise there is, the more sharper alphas reflect the stock's loading on the measurement error ($\beta_{\xi,n}$).

Fortunately, there is an easy way to verify that the sharper alphas are not driven by this omitted variable bias. The key idea is to validate sharper alphas by comparing them to OLS alphas, which are not affected by this bias. Even though OLS alphas are unreliable at the stock level, we can still obtain decently stable estimates at the portfolio level because some of the idiosyncratic noise is already diversified away. We can therefore use portfolio OLS alphas as a benchmark to validate our sharper alphas. If the sharper alphas can match the cross-sectional patterns generated by the OLS alphas on characteristic sorted portfolios, we can mitigate concerns that this omitted variable bias is driving our results.

In next section, we find that we can indeed rule out the omitted variable bias, and that the estimated tangency can therefore produce meaningful estimates of sharper alphas.

3 Empirical Results

In this section, we first use the classic characteristic-sorted portfolios to validate our sharper alpha $\hat{\alpha}^*$ and rule out potential problems with measurement errors in the tangency factor. We then apply our new measure to single stocks and show that they provide statistical power where the alphas do not.

3.1 Sharper alpha on sorted portfolios

We compare sharper alphas and OLS alphas on the 10 value sorted Fama-French portfolios. We plot the alphas with respect to the CAPM, Fama-French 3 factor (FF3) (in Figure (8)), q-factor, and Fama-French 5 factor (FF5) models (in Figure (9)). The error bars mark the 95% confidence intervals.

We make a number of reassuring observations. The point estimates of sharper alphas follow the same cross-sectional patterns as the OLS alphas, which helps to rule out the potential issue where measurement error is driving the cross-sectional variation in sharper alphas. Further, the sharper alphas track the levels of OLS alphas. We see in Figure (8) , for example, that the value-sorted portfolios earn between -2.3% to 2.5% with respect to the CAPM, when the sharper alphas were estimated to be between -2.4% and 2.5%. The similarities are signs that even if there is some attenuation bias, it is fairly limited.

The sharper alpha provide approximately a ten fold reduction in the standard errors, compared to traditional CAPM alphas. This effect becomes even more significant as we move to smaller portfolios, where less of the idiosyncratic noise is diversified away. In Figure (12) for example, we see that this variance reduction is especially large when we consider the much smaller 10x10 value and size portfolios.

In the appendix, we report traditional and Sharper alphas with respect to portfolios sorted on beta, earnings-to-price, investment-to-asset, momentum and return on equity. Using Sharper alphas provides similar efficiency gains compared to the value and size anomalies.

3.2 Sharper alpha on single stocks

We now apply sharper alphas to individual stocks to show that they can detect significant alpha without sacrificing cross-sectional variability. We estimate alphas on 5 years of daily returns from 2014-12-31 to 2019-12-31. We use a 5 year window (similar to Fama and MacBeth 1973) as a reasonable compromise between having more relevant ex-ante characteristics and having a longer sample of ex-poste realized returns.

In Figure (3), we provide kernel density plots for the cross-section of estimated OLS and sharper alphas. While most OLS alphas are not statistically significant, many of the sharper alphas are. Using a t-statistic of 3 as the critical value, 27.88% of sharper alphas are statistically significant while only 0.12% of the OLS alphas are. We plot the percentage of statistically significant alphas at different critical values in Figure (4) of the appendix.

We plot these alphas against stock characteristics to see if sharper alphas can in fact detect the portfolio patterns at the stock level. We focus on the betting-against-beta, investment, and profitability anomalies, which generated significant alphas over this sample period. With the betting-against-beta (Figure (6)) and investment (Figure (7)) anomalies, we see statistically significant patterns on the sharper alphas but not for OLS alphas. The cross-sectional relations we observe are consistent with the portfolio-level results on quintile-sorted portfolios (show in red). Sharper alphas therefore provide statistically significant alphas variation without sacrificing stock-level variability in characteristics.

For the profitability anomaly (Figure (5)), the sharper alphas on individual stocks reveal additional details that are too noisy to see in OLS alphas. The stock level relation between sharper alphas and profitability mirrors the portfolio level results, on average, but has two distinct components. The overall positive alpha-profitability relation is an average between a strong positive relation amongst negative profitability stocks and a much milder relation amongst positive profitability stocks.

This insight from the profitability anomaly provides new guidance for theory. An unprofitable firm only has value because it could become profitable in the future. In other words, the firm's valuation is largely driven by its growth options. Since a large part of the profitability alpha seems to be driven by unprofitable firms, we should investigate how these real-options affect a firm's decisions and expected returns (as in Ericsson, Jo, and Lotfaliei 2020).

4 Conclusion

This paper develops a new approach to measuring abnormal returns. This approach uses the tangency portfolio as a "better yard stick" for measuring alphas. We show that the resulting estimates are robust against measurement error and provide statistical power even when the OLS alphas fail.

Sharper alphas provide several benefits. First they help reduce estimation noise and provide statistically significant alphas without forming portfolios and incurring aggregations errors. Second, by applying sharper alphas to individual stocks, we find cross-sectional patterns that are not visible at the portfolio-level or from the noisier OLS alphas. Third, sharper alphas can lend their efficiency gains to existing alpha-based tests. We illustrate this feature by providing the sharper alpha equivalent of the GRS test.

We hope that our approach will provide an improved tool-kit for empirical tests in asset pricing, and that future research will use sharper alphas to sharpen existing, alpha-based tests. In addition, since we relate the true alphas to a decomposition of betas, another promising application is to develop tests that go beyond alpha and derive testable implications on the joint cross-sectional distribution of alphas and betas. We leave these applications for future research.

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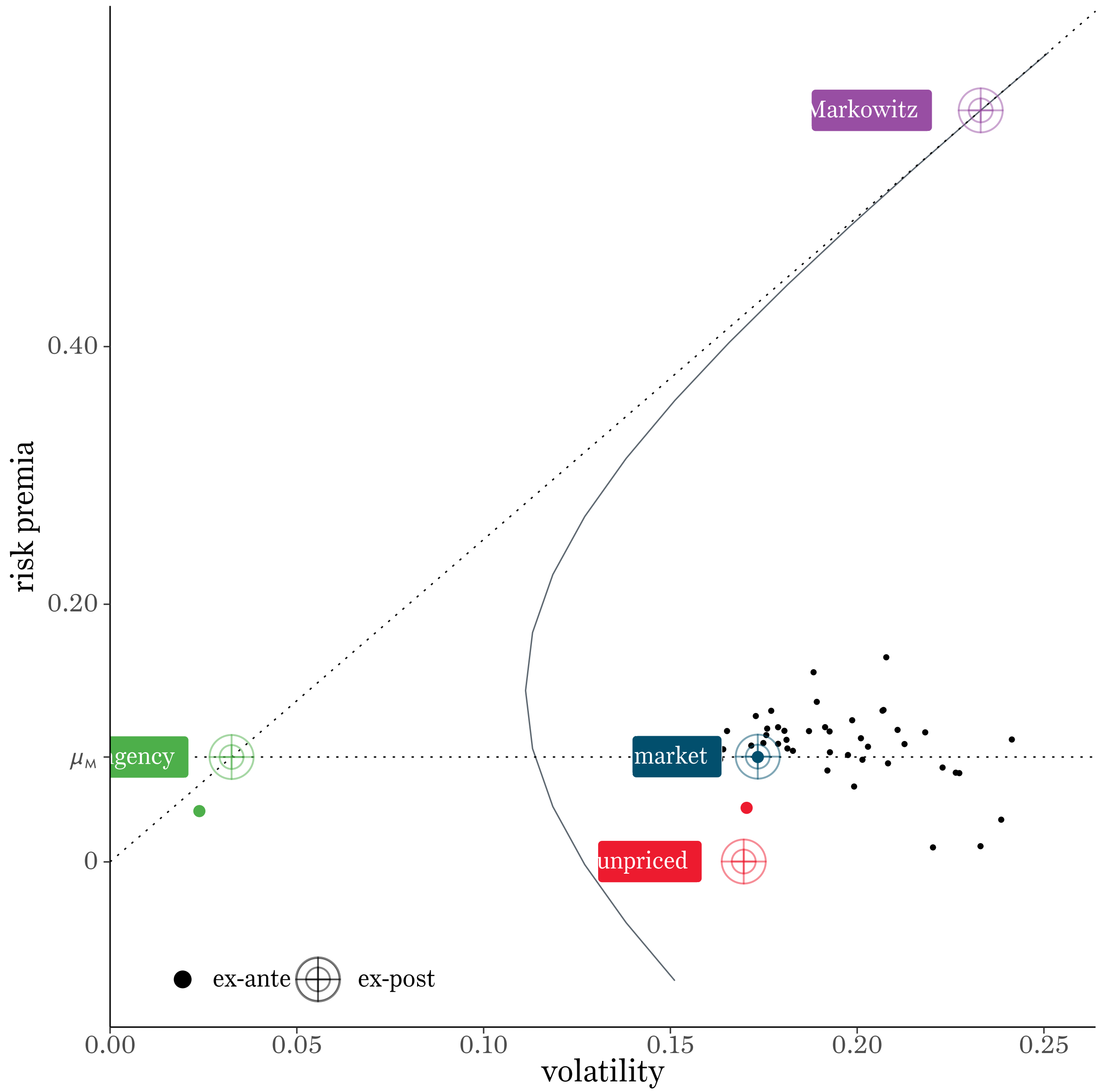

Figure 1: **Tangency and Unpriced Portfolios: ex-ante estimates vs ex-post realizations.** This figure plots the risk premia and volatilities estimated with 30 years of daily returns on 2019-12-31. The 25 Fama-French portfolios, 10 momentum portfolios, and 18 q-factor portfolios are shown as black dots. Coloured dots show the market portfolio and ex-ante estimates of its tangency and unpriced components. The in-sample, ex-poste realized realized tangency and unpriced portfolios are shown as cross-hairs.

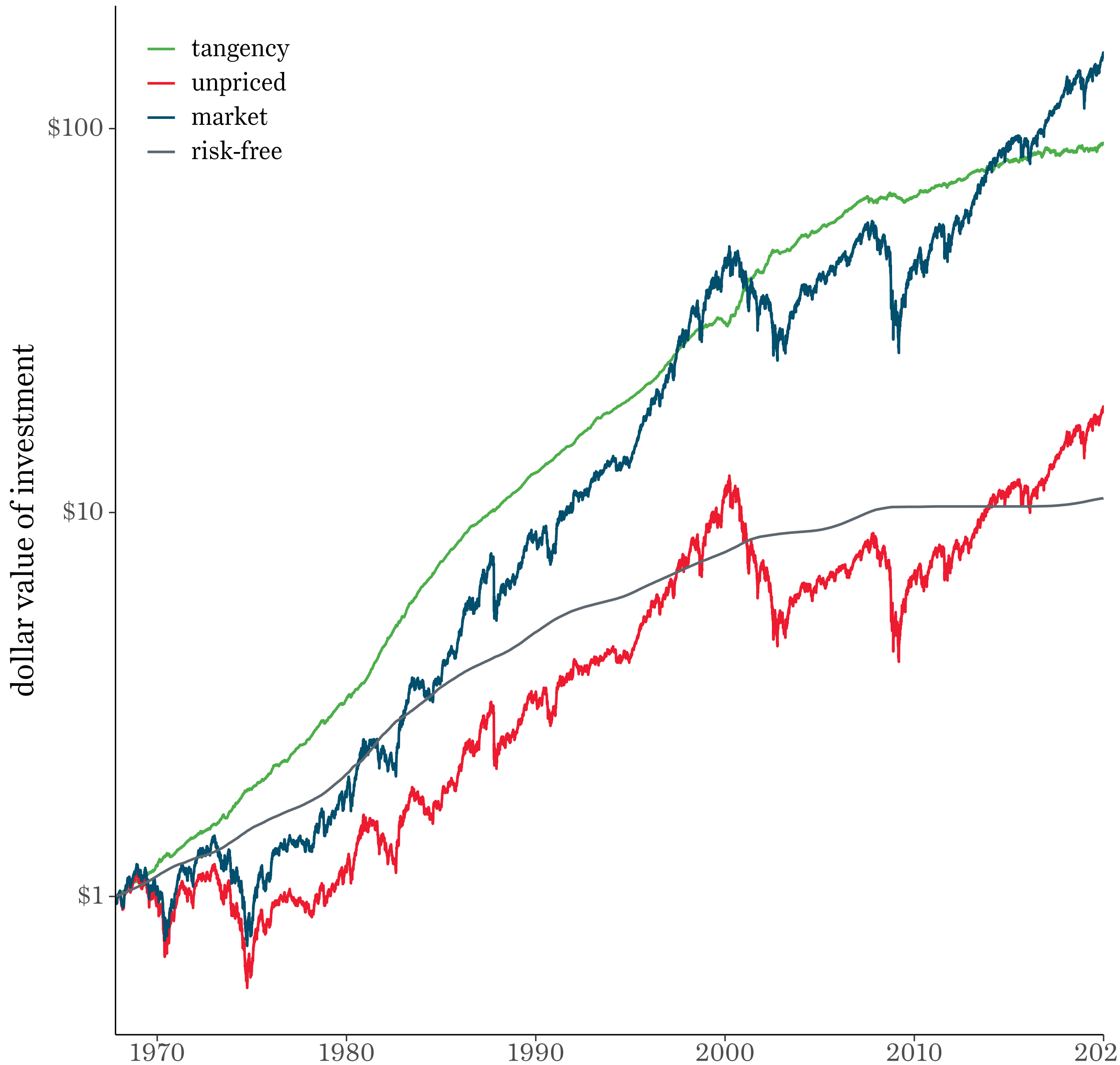

Figure 2: **Decomposition of Market Returns.** This figure plots the cumulative log returns to four portfolios: 1) the tangency portfolio, 2) the unpriced portfolio, 3) the value-weighted market index, and 4) the risk-free asset. The tangency portfolio earns the market risk premium at less than a third of the market's volatility. The unpriced portfolio generates the remaining volatility in market returns, while earning zero risk-premium.

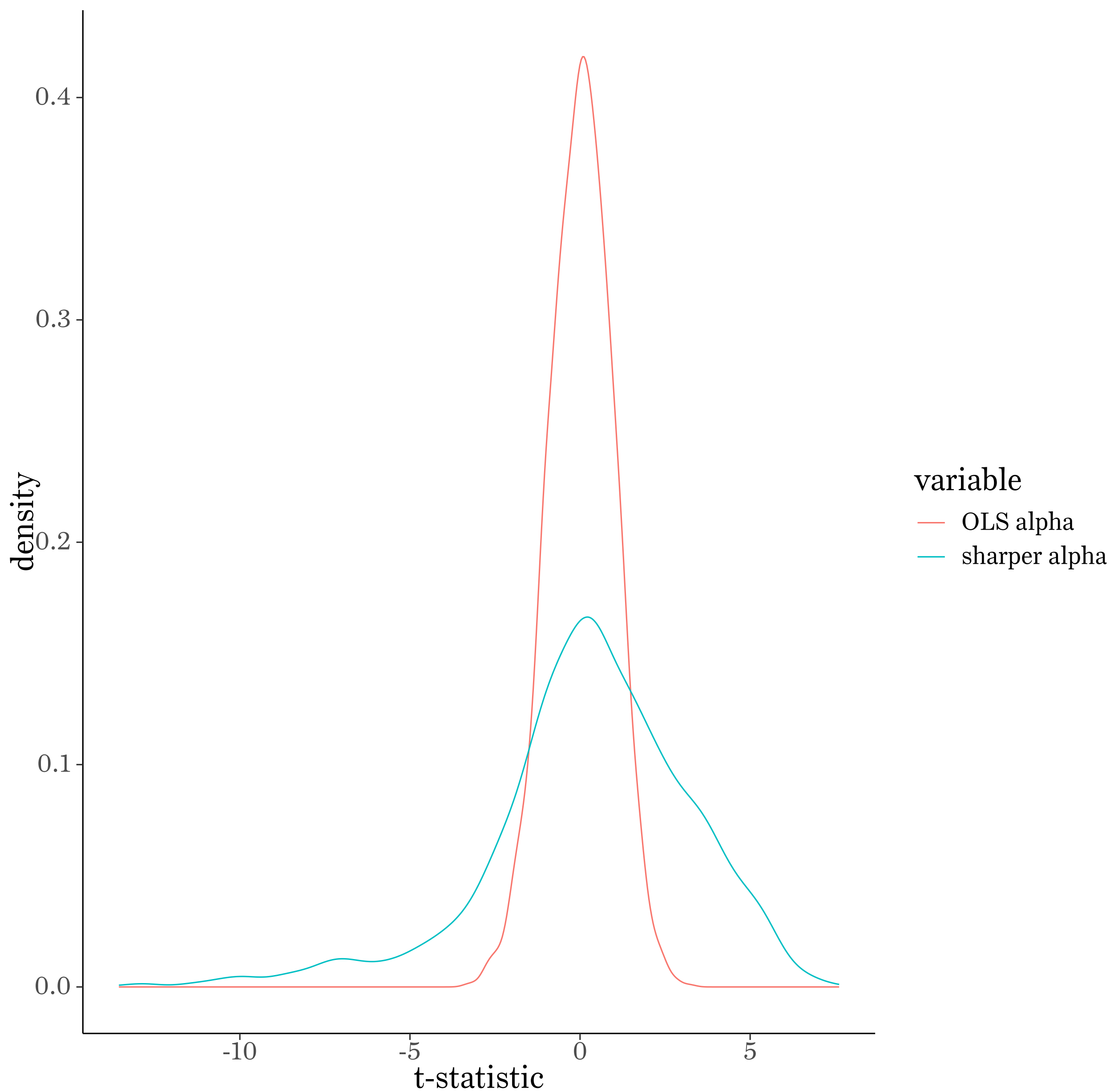

Figure 3: **Kernel density of the cross-section of estimated alphas.** This figure plots cross-sectional distribution of alphas estimated on the daily U.S. stock returns between 2014-12-31 and 2019-12-31.

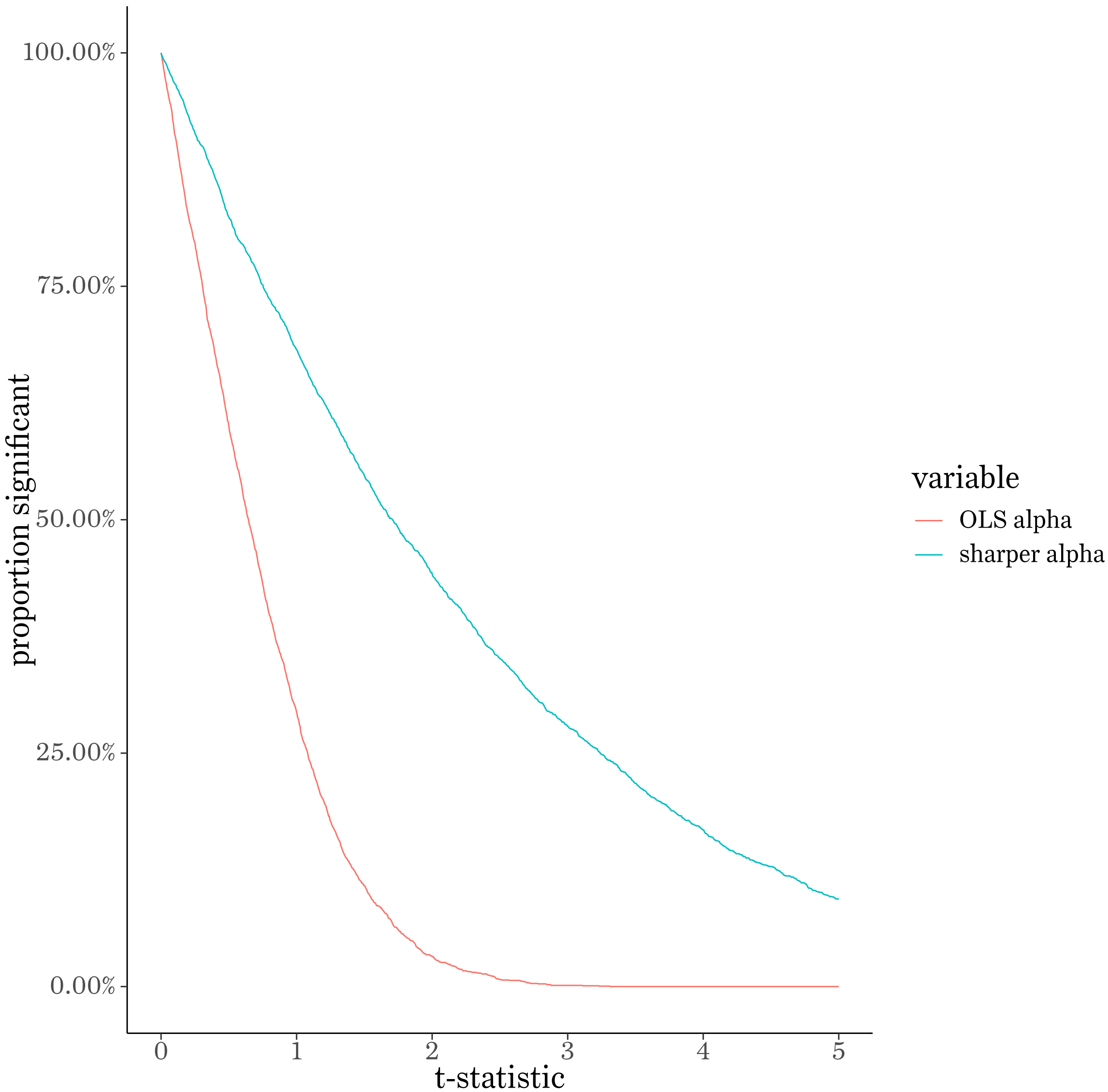

Figure 4: **Proportion of significant alphas.** We plot the proportion of alphas that are significant at different critical values. Alphas are estimated on daily U.S. stock returns from 2014-12-31 to 2019-12-31.

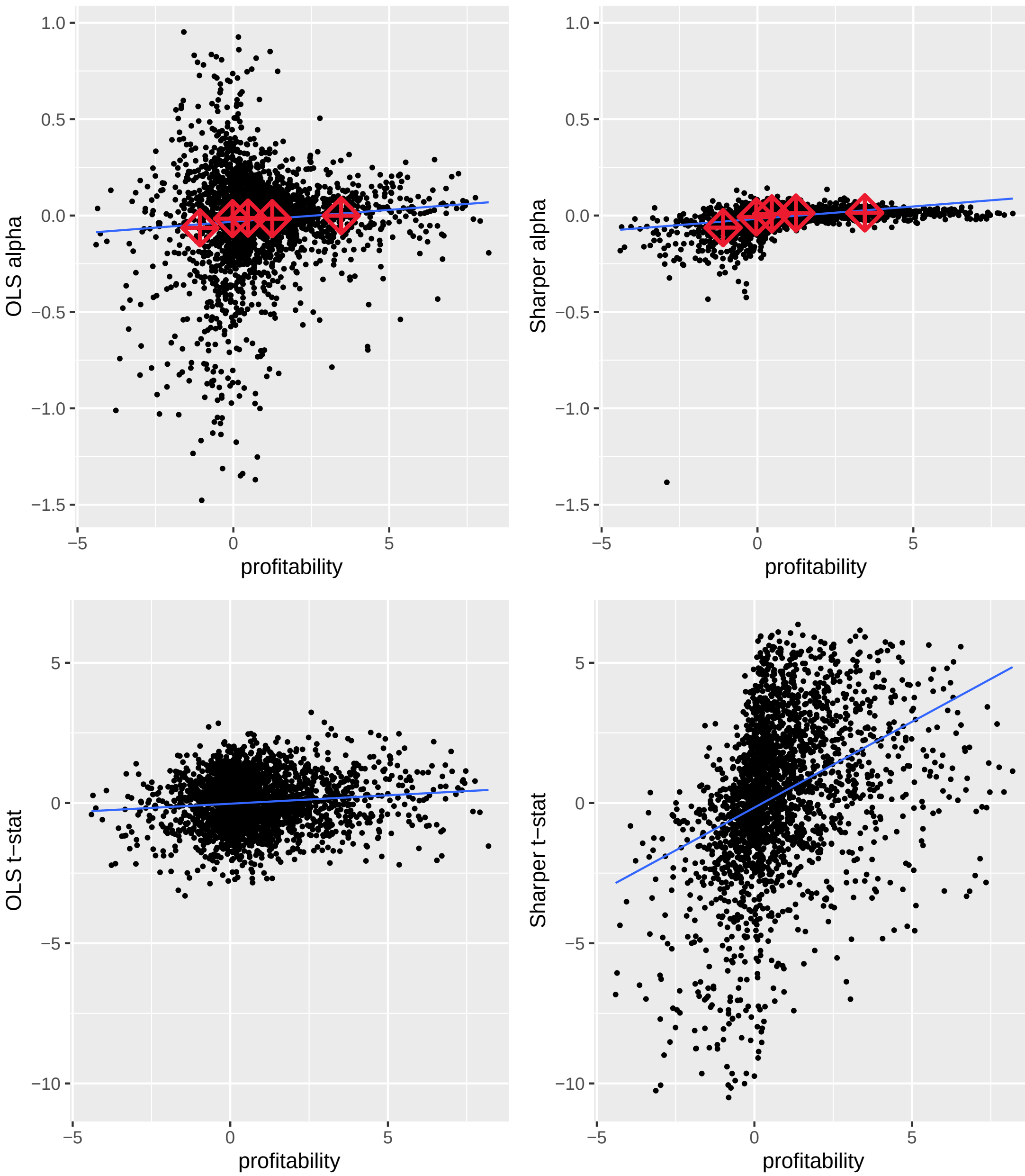

Figure 5: **The profitibility anomaly: ols alpha vs sharper alpha.** This figure plots the sharper and OLS alphas using daily returns from 2014-12-31 to 2019-12-31, against their profitability (as measured by their average ROE from 2004 to 2014). The quintile-sorted portfolios are shown in red. The average cross-sectional relation and the portfolio-level relation are similar whether we look at sharper alphas or OLS alphas. At the stock level, however, sharper alphas reveal a distinct, nonlinear relation with negative profitability stocks that is too noisy to see with traditional OLS alphas.

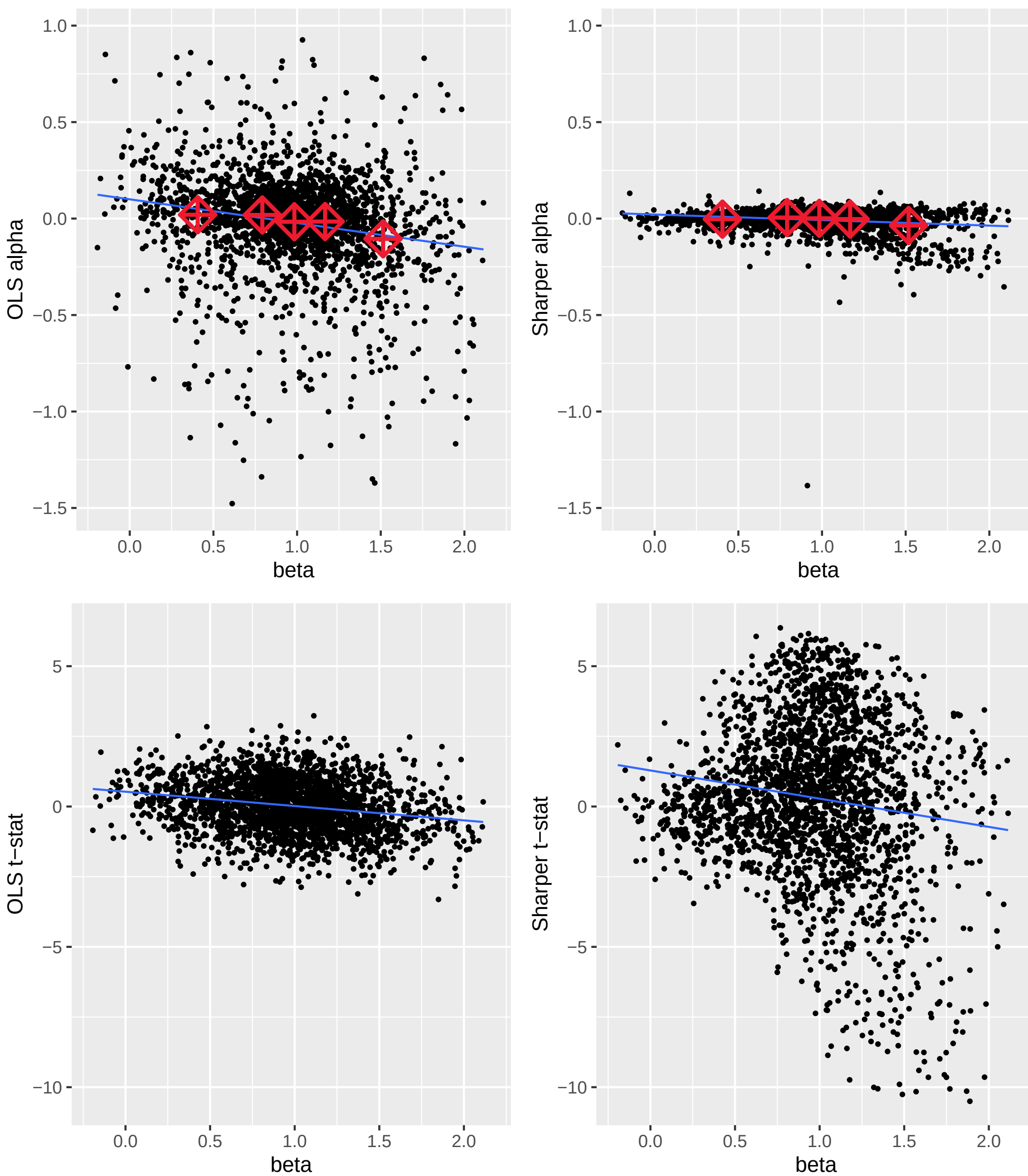

Figure 6: **The betting-against-beta anomaly: ols alpha vs sharper alpha.** This figure plots the sharper and OLS alphas using daily returns from 2014-12-31 to 2019-12-31, against their betas. The quintile-sorted portfolios are shown in red. The average cross-sectional relation and the portfolio-level relation are similar whether we look at sharper alphas or OLS alphas. At the stock level, sharper alphas provide significant noise reduction.

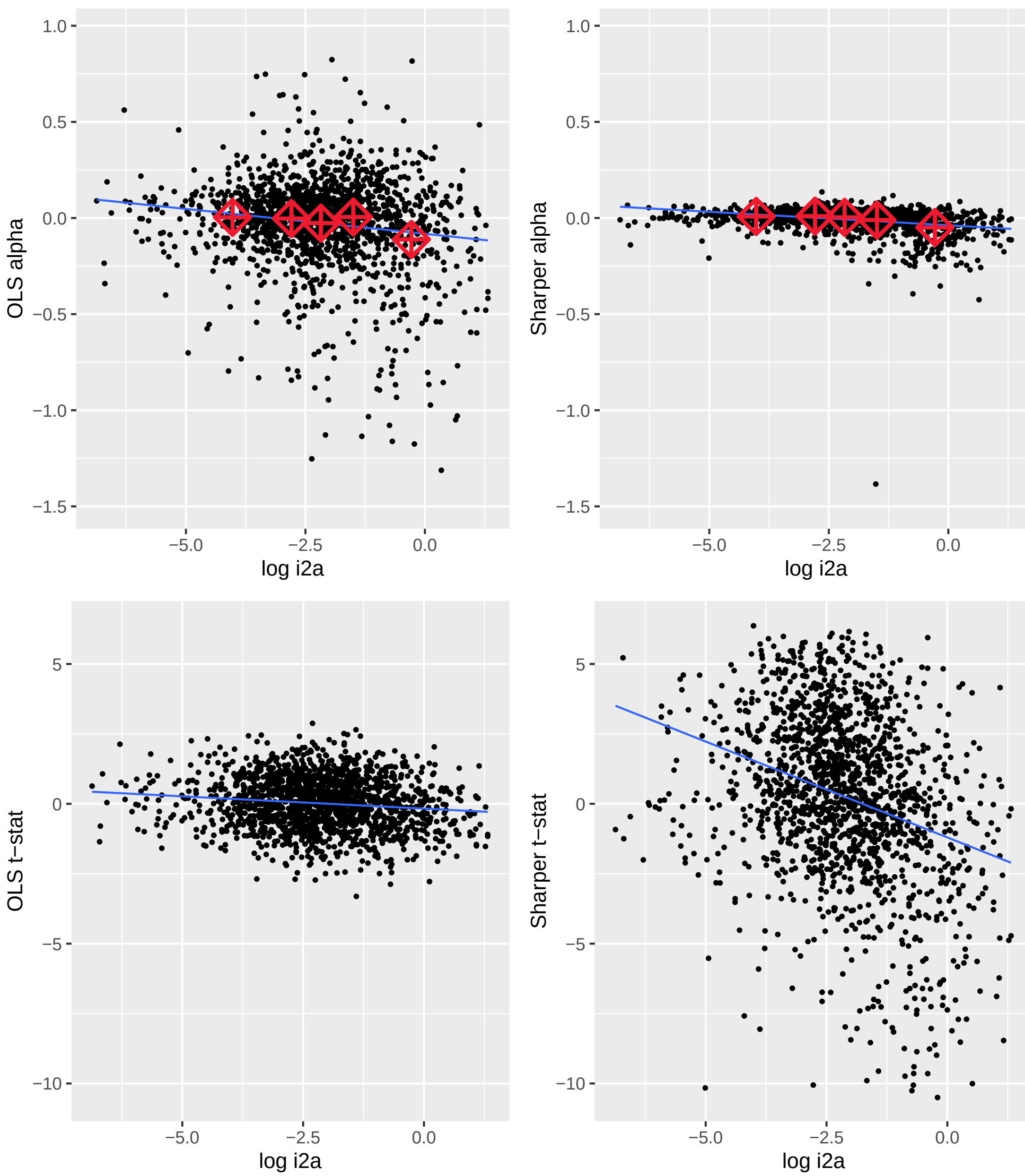

Figure 7: **The investment anomaly: ols alpha vs sharper alpha.** This figure plots the sharper and OLS alphas using daily returns from 2014-12-31 to 2019-12-31, against their investment-to-asset ratio (averaged over the 10 prior years from 2004 to 2014). The quintile-sorted portfolios are shown in red. The average cross-sectional relation and the portfolio-level relation are similar whether we look at sharper alphas or OLS alphas. At the stock level, sharper alphas provide significant noise reduction.

	r_M (1)	r_τ (2)	r_U (3)
Sharpe ratio	0.400***	1.911***	0.145
	(0.138)	(0.138)	(0.138)
risk premia	0.064***	0.041***	0.023
	(0.022)	(0.003)	(0.022)
Note:	*p<0.1; **p<0.05; ***p<0.01		

Table 1: This table presents the Sharpe ratios of the tangency, unpriced, and market portfolios, estimated with daily returns from 1967 to 2019.

	r_M (1)	r_M (2)	r_M (3)
r_τ	1.000***	1.033***	
	(0.000)	(0.033)	
r_U	1.000***		1.002***
	(0.000)		(0.00002)
Constant	-0.000	0.0001	0.0002***
	(0.000)	(0.0001)	(0.00002)
R^2	1.00	0.066	0.938
Note:	*p<0.1; **p<0.05; ***p<0.01		

Table 2: This table presents the coefficients and R^2 from regressing daily market excess returns from 1967 to July 2019 on its tangency and unpriced components.

	r_M	r_τ	r_U
r_M	100.00%	15.58%	99.10%
r_τ	15.58%	100.00%	2.20%
$r_U = r_M - r_\tau$	99.10%	2.20%	100.00%

Table 3: This table presents the correlations between the market excess returns and its components, estimated on daily returns from 1967 to 2019.

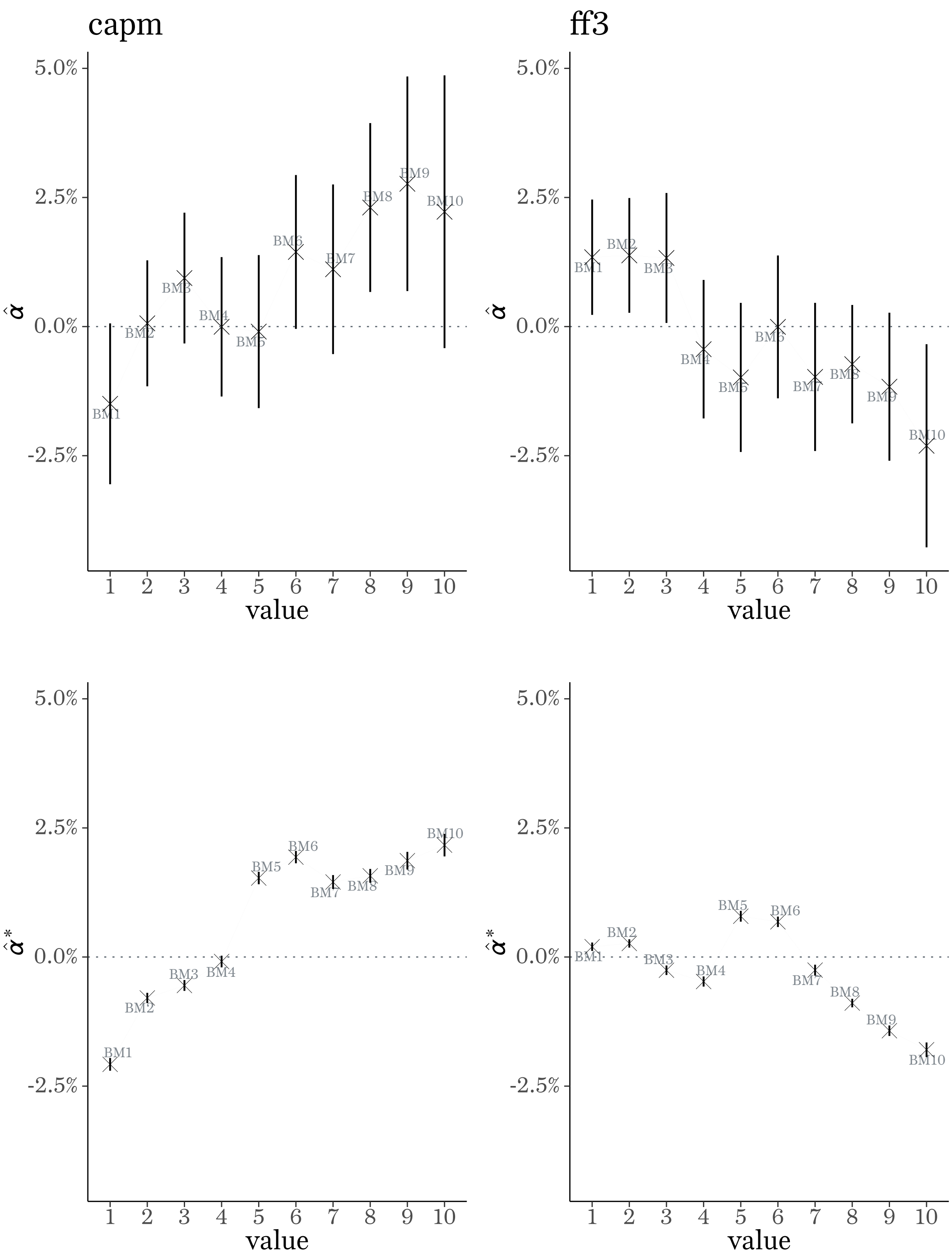

Figure 8: **Cross-Sectional Alpha.** This figure shows the CAPM and FF3 alphas for the 10 value sorted portfolios, estimated from daily returns between 1967 to 2019. The error bars indicate the 95% confidence intervals.

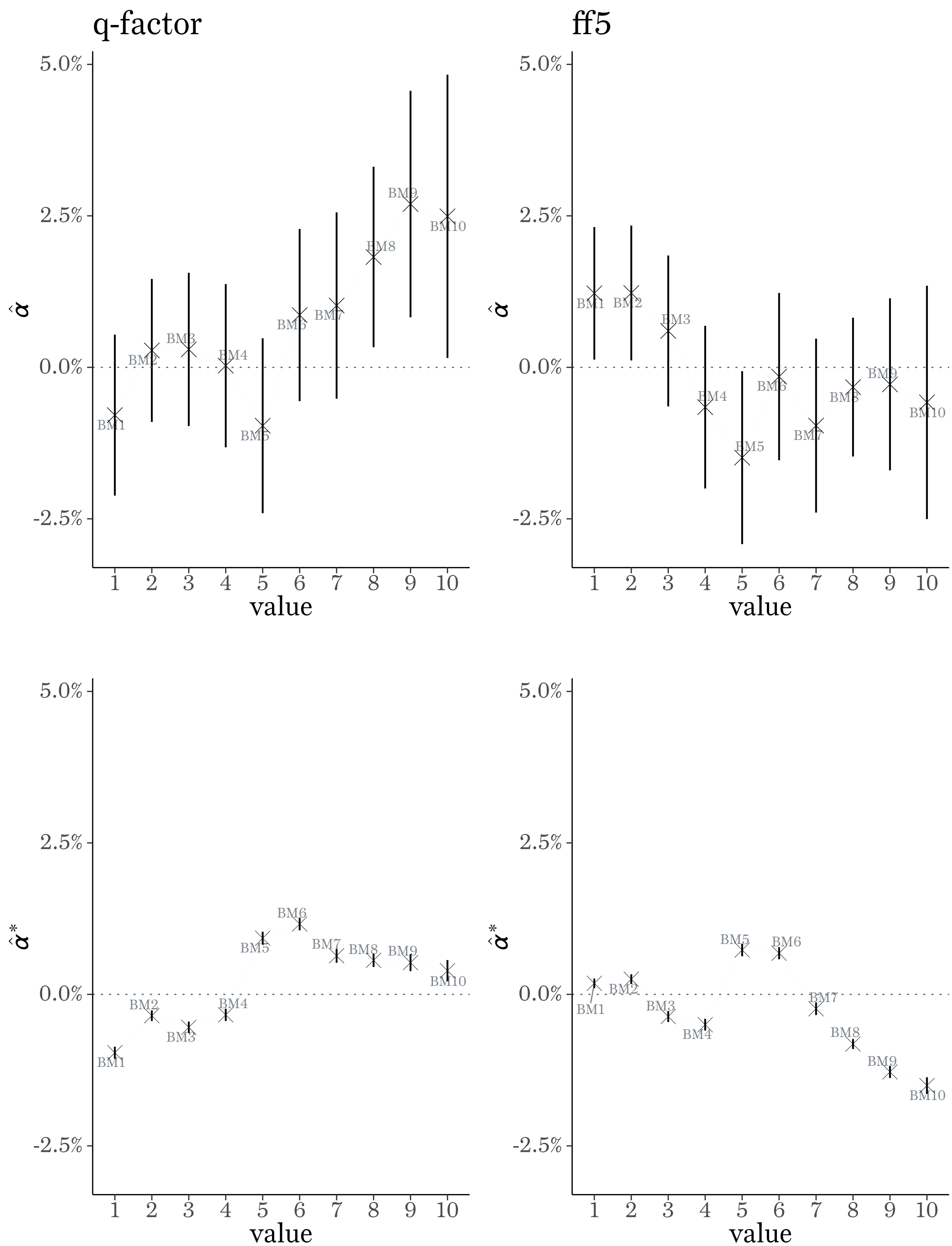

Figure 9: **Cross-Sectional Alpha.** This figure shows the q-Factor and FF5 alphas for the 10 value sorted portfolios, estimated from daily returns between 1967 to 2019. The error bars indicate the 95% confidence intervals.

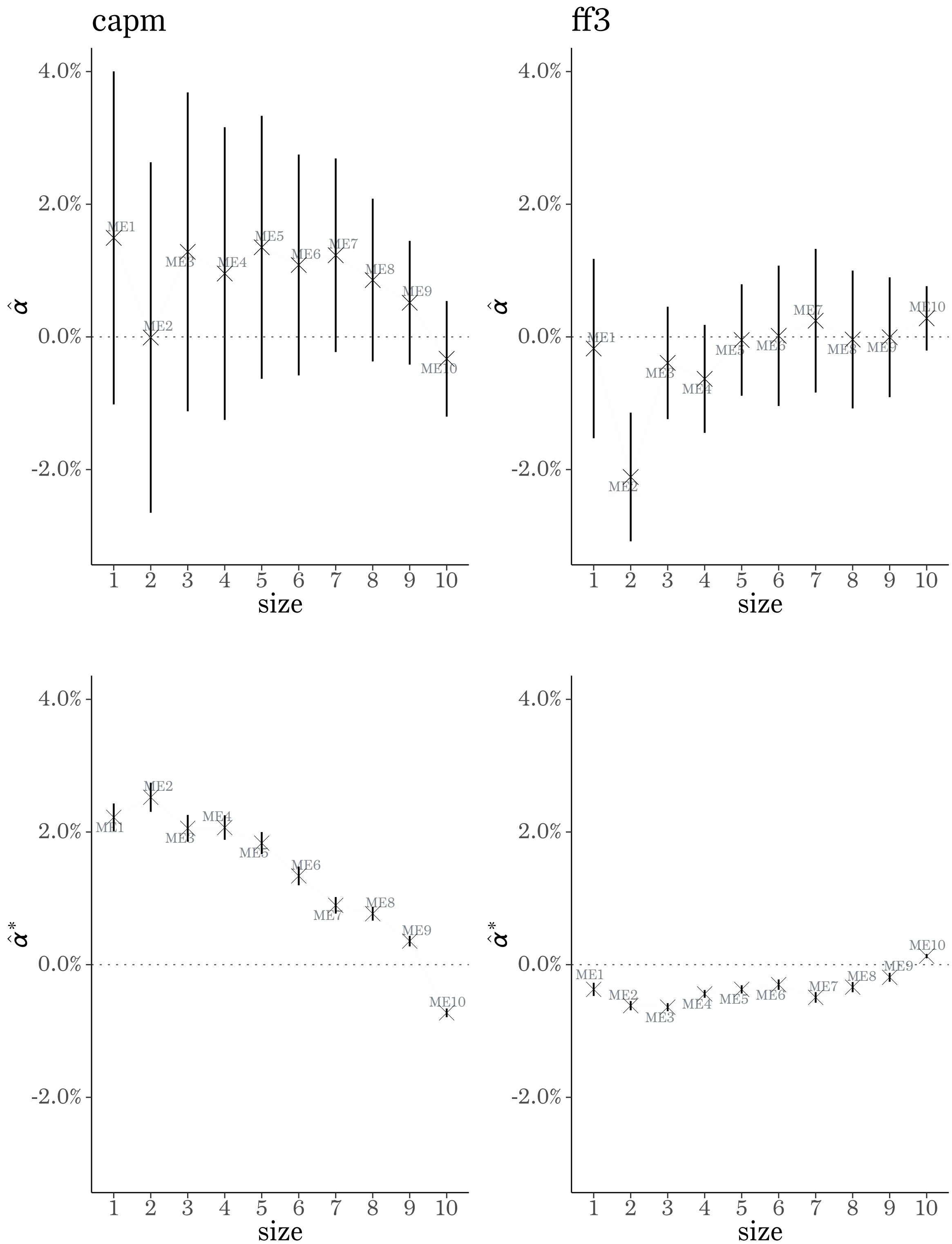

Figure 10: **Cross-Sectional Alpha.** This figure shows the CAPM and FF3 alphas for the 10 size sorted portfolios, estimated from daily returns between 1967 to 2019. The error bars indicate the 95% confidence intervals.

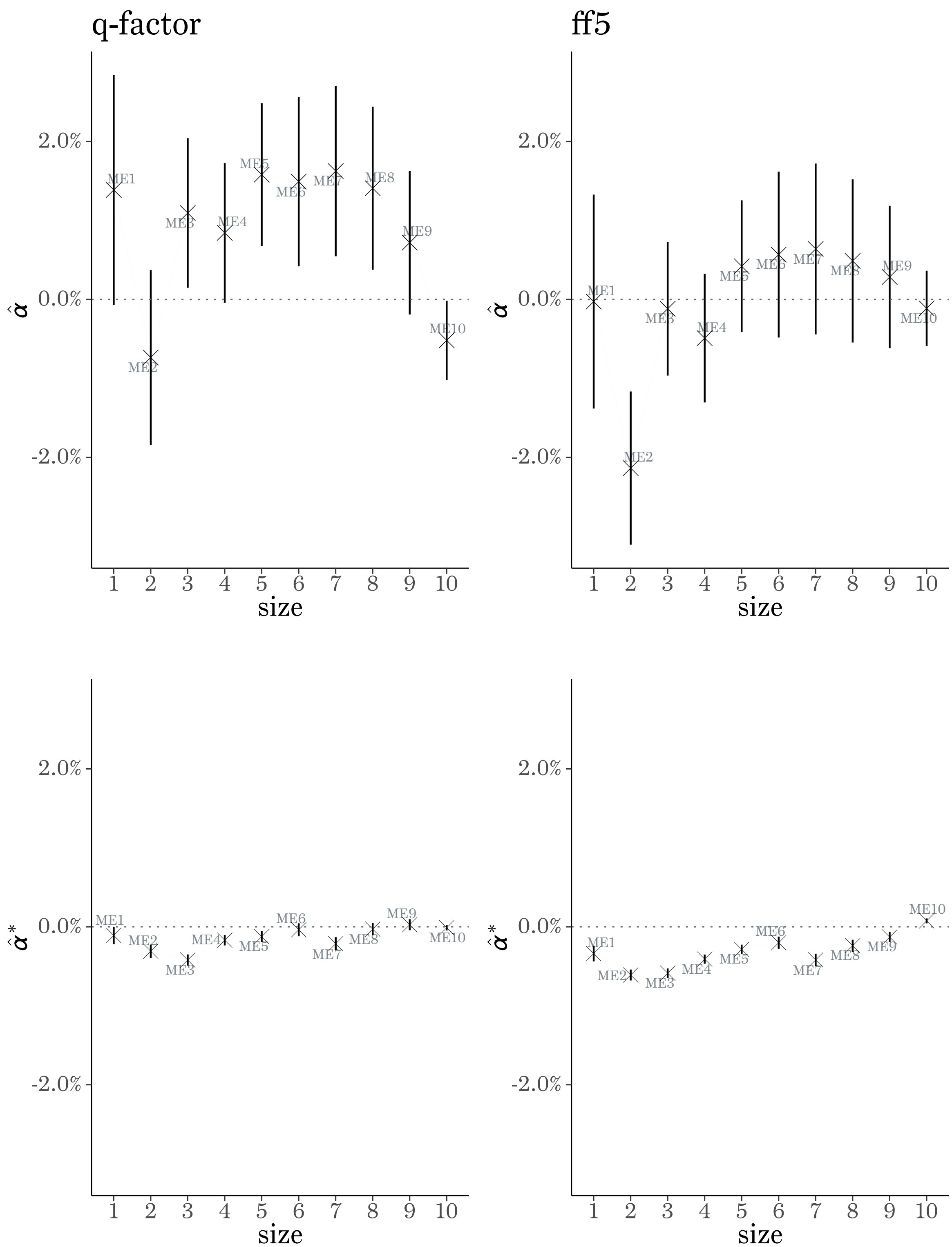

Figure 11: **Cross-Sectional Alpha.** This figure shows the q-Factor and FF5 alphas for the 10 size sorted portfolios, estimated from daily returns between 1967 to 2019. The error bars indicate the 95% confidence intervals.

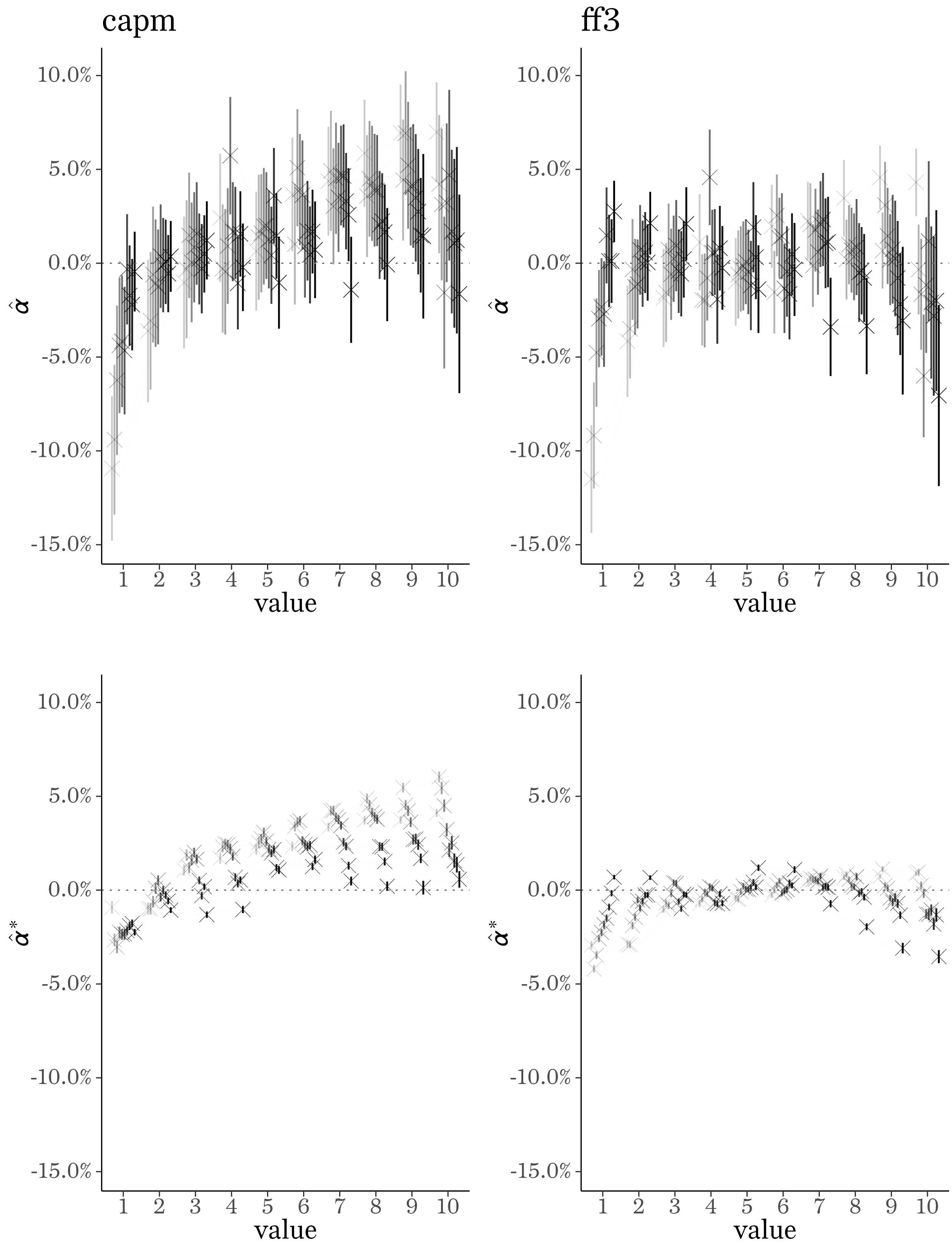

Figure 12: **Cross-Sectional Alpha.** This figure shows the CAPM and FF3 factor alphas for the 10x10 value and size sorted portfolios, estimated from daily returns between 1967 to 2019. The error bars indicate the 95% confidence intervals.

A Distribution of estimators

A.1 Alpha vs Sharper Alpha

A.1.1 Alpha

Take Y the vector of T observations of excess returns and $X = [\iota | r_{\mathsf{M}}]$ the $T \times 2$ matrix containing the constant vector and the vector of market excess returns, we can write the returns as

$$Y = X\gamma + u \tag{52}$$

$$\gamma = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \tag{53}$$

where γ has the CAPM coefficients and $\underset{T\times 1}{u}$ is the T vector of CAPM residuals.

The sample estimates for the coefficient is

$$\begin{aligned} \hat{\gamma} &= \left(X^{\intercal} X\right)^{-1} X^{\intercal} Y \\ &= \gamma + \left(X^{\intercal} X\right)^{-1} X^{\intercal} u \end{aligned}$$

and has a variance of

$$\text{VAR}\left[\hat{\gamma}\right] = \left(X^{\intercal} X\right)^{-1} X^{\intercal} \mathrm{E}\left[u u^{\intercal}\right] X \left(X^{\intercal} X\right)^{-1}$$

. Assuming the errors are uncorrelated and homoscedastic over time, we have

$$\begin{aligned}\text{VAR}\left[\hat{\gamma}\right] &= \left(X^{\intercal}X\right)^{-1}\cdot\sigma_u^2\\ \sigma_u^2 &= \sigma_\varepsilon^2 + \left(\beta_{\tau,n}-\beta_{\mathsf{U},n}\right)^2\frac{\sigma_{\mathsf{U}}^2\sigma_\tau^2}{\sigma_{\mathsf{M}}^2}\end{aligned}$$

. Note that

$$\begin{aligned}\left(X^{\intercal}X\right)^{-1} &= \begin{bmatrix} x_2^{\intercal}x_2 & -x_1^{\intercal}x_2\\ -x_1^{\intercal}x_2 & x_1^{\intercal}x_1\end{bmatrix}\cdot\frac{1}{x_1^{\intercal}x_1x_2^{\intercal}x_2-\left(x_1^{\intercal}x_2\right)^2}\\ &= \begin{bmatrix}\sum r_{\mathsf{M}}^2 & -\sum r_{\mathsf{M}}\\ -\sum r_{\mathsf{M}} & T\end{bmatrix}\cdot\frac{1}{T\cdot\sum r_{\mathsf{M}}^2-\left(\sum r_{\mathsf{M}}\right)^2}\\ \text{VAR}\left[\hat{\alpha}_n\right] &= \frac{\sum r_{\mathsf{M}}^2}{T\cdot\sum r_{\mathsf{M}}^2-\left(\sum r_{\mathsf{M}}\right)^2}\left(\sigma_\varepsilon^2+\left(\beta_{\tau,n}-\beta_{\mathsf{U},n}\right)^2\frac{\sigma_{\mathsf{U}}^2\sigma_\tau^2}{\sigma_{\mathsf{M}}^2}\right)\\ &= \frac{T\cdot\bar{r}_{\mathsf{M}}^2+\sum\left(r_{\mathsf{M}}^2-\bar{r}_{\mathsf{M}}\right)}{T\cdot\sum\left(r_{\mathsf{M}}^2-\bar{r}_{\mathsf{M}}\right)}\left(\sigma_\varepsilon^2+\left(\beta_{\tau,n}-\beta_{\mathsf{U},n}\right)^2\frac{\sigma_{\mathsf{U}}^2\sigma_\tau^2}{\sigma_{\mathsf{M}}^2}\right)\\ &= T^{-1}\left(\bar{r}_{\mathsf{M}}^2\left(\frac{\sum\left(r_{\mathsf{M}}^2-\bar{r}_{\mathsf{M}}\right)}{T}\right)^{-1}+1\right)\left(\sigma_\varepsilon^2+\left(\beta_{\tau,n}-\beta_{\mathsf{U},n}\right)^2\frac{\sigma_{\mathsf{U}}^2\sigma_\tau^2}{\sigma_{\mathsf{M}}^2}\right)\end{aligned}$$

Defining the consistent estimators

$$\hat{\sigma}_{\mathsf{M}}^2 = \frac{\sum\left(r_{\mathsf{M}}^2-\bar{r}_{\mathsf{M}}\right)}{T} \tag{54}$$

$$\bar{r}_{\mathsf{M}} = \frac{\sum r_{\mathsf{M}}}{T} \tag{55}$$

then we can rewrite the variance of the alpha estimator as

$$\text{VAR}\left[\hat{\alpha}_n\right] = T^{-1}\left(\frac{\bar{r}_{\mathsf{M}}^2}{\hat{\sigma}_{\mathsf{M}}^2}+1\right)\left(\sigma_\varepsilon^2+\left(\beta_{\tau,n}-\beta_{\mathsf{U},n}\right)^2\frac{\sigma_{\mathsf{U}}^2\sigma_\tau^2}{\sigma_{\mathsf{M}}^2}\right) \tag{56}$$

A.1.2 Sharper Alpha

Take Y the vector of T observations of excess returns and $X = [r_\tau | r_{\text{u}}]$, we can write the returns as

$$Y = X\gamma + \varepsilon \tag{57}$$

$$\gamma = \begin{bmatrix} \beta_{\tau,n} \\ \beta_{\text{u},n} \end{bmatrix} \tag{58}$$

.

The sample estimates for the coefficient is

$$\begin{aligned} \hat{\gamma} &= \left(X^\intercal X\right)^{-1} X^\intercal Y \\ &= \gamma + \left(X^\intercal X\right)^{-1} X^\intercal \varepsilon \end{aligned}$$

and has a variance of

$$\text{VAR}\left[\hat{\gamma}\right] = \left(X^\intercal X\right)^{-1} X^\intercal \mathrm{E}\left[\varepsilon\varepsilon^\intercal\right] X \left(X^\intercal X\right)^{-1}$$

. Assuming the errors are uncorrelated and homoscedastic, we have

$$\text{VAR}\left[\hat{\gamma}\right] = \left(X^\intercal X\right)^{-1} \cdot \sigma_\varepsilon^2$$

. Note that

$$\left(X^\intercal X\right)^{-1} = \begin{bmatrix} x_2^\intercal x_2 & -x_1^\intercal x_2 \\ -x_1^\intercal x_2 & x_1^\intercal x_1 \end{bmatrix} \cdot \frac{1}{x_1^\intercal x_1 x_2^\intercal x_2 - \left(x_1^\intercal x_2\right)^2}$$

$$= \begin{bmatrix} r_{\mathsf{U}}^\intercal r_{\mathsf{U}} & -r_\tau^\intercal r_{\mathsf{U}} \\ -r_\tau^\intercal r_{\mathsf{U}} & r_\tau^\intercal r_\tau \end{bmatrix} \cdot \frac{1}{r_\tau^\intercal r_\tau \cdot r_{\mathsf{U}}^\intercal r_{\mathsf{U}} - \left(r_\tau^\intercal r_{\mathsf{U}}\right)^2}$$

$$\textsc{var}\left[\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}\right] = \begin{bmatrix} 1 & -1 \end{bmatrix} \textsc{var}\left[\hat{\gamma}\right] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{\left(r_{\mathsf{U}}^\intercal r_{\mathsf{U}} + 2 r_\tau^\intercal r_{\mathsf{U}} + r_\tau^\intercal r_\tau\right) \sigma_\varepsilon^2}{r_\tau^\intercal r_\tau \cdot r_{\mathsf{U}}^\intercal r_{\mathsf{U}} - \left(r_\tau^\intercal r_{\mathsf{U}}\right)^2}$$

$$= \frac{r_{\mathsf{M}}^\intercal r_{\mathsf{M}} \sigma_\varepsilon^2}{r_\tau^\intercal r_\tau \cdot r_{\mathsf{U}}^\intercal r_{\mathsf{U}} - \left(r_\tau^\intercal r_{\mathsf{U}}\right)^2}$$

$$= T^{-1} \frac{T^{-1} r_{\mathsf{M}}^\intercal r_{\mathsf{M}} \sigma_\varepsilon^2}{T^{-1} r_\tau^\intercal r_\tau \cdot T^{-1} r_{\mathsf{U}}^\intercal r_{\mathsf{U}} - \left(T^{-1} r_\tau^\intercal r_{\mathsf{U}}\right)^2}$$

$$= T^{-1} \frac{\bar{r}_{\mathsf{M}}^2 + \hat{\sigma}_{\mathsf{M}}^2}{\left(\bar{r}_\tau^2 + \hat{\sigma}_\tau^2\right) \cdot \left(\bar{r}_{\mathsf{U}}^2 + \hat{\sigma}_{\mathsf{U}}^2\right) - \left(\bar{r}_\tau \bar{r}_{\mathsf{U}} + \hat{\sigma}_{\tau,\mathsf{U}}^2\right)^2} \sigma_\varepsilon^2$$

A.1.3 Variance Ratio

Asymptotically, as $T \to \infty$ the probability limits of the cross-products converge to the second moments

$$\hat{\sigma}^2_{\mathsf{M}} \to \sigma^2_{\mathsf{M}} \tag{59}$$

$$\bar{r}_{\mathsf{M}} \to \mu_{\mathsf{M}} \tag{60}$$

$$\hat{\sigma}^2_{\tau} \to \sigma^2_{\tau} \tag{61}$$

$$\bar{r}_{\tau} \to \mu_{\mathsf{M}} \tag{62}$$

$$\hat{\sigma}^2_{\mathsf{U}} \to \sigma^2_{\mathsf{U}} \tag{63}$$

$$\hat{\sigma}^2_{\tau,\mathsf{U}} \to \sigma_{\tau,\mathsf{U}} = 0 \tag{64}$$

$$T\,\text{VAR}\left[\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}\right] \to \left(\sigma^2_{\mathsf{M}} + \mu^2_{\mathsf{M}}\right) \frac{\sigma^2_{\varepsilon}}{\left(\sigma^2_{\tau} + \mu^2_{\mathsf{M}}\right) \cdot \sigma^2_{\mathsf{U}}} \tag{65}$$

$$T\,\text{VAR}\left[\bar{r}_{\tau} \frac{\hat{\sigma}^2_{\mathsf{U}}}{\hat{\sigma}^2_{\mathsf{M}}} \left(\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}\right)\right] \to \frac{\mu^2_{\mathsf{M}}}{\sigma^2_{\tau} + \mu^2_{\mathsf{M}}} \cdot \frac{\sigma^2_{\mathsf{U}}}{\sigma^2_{\mathsf{M}}} \cdot \frac{\sigma^2_{\mathsf{M}} + \mu^2_{\mathsf{M}}}{\sigma^2_{\mathsf{M}}} \cdot \sigma^2_{\varepsilon} \tag{66}$$

$$T\,\text{VAR}\left[\hat{\alpha}_n\right] \to \left(\sigma^2_{\mathsf{M}} + \mu^2_{\mathsf{M}}\right) \left(\left(\beta_{\tau,n} - \beta_{\mathsf{U},n}\right)^2 \frac{\sigma^2_{\tau}}{\sigma^2_{\mathsf{M}}} \frac{\sigma^2_{\mathsf{U}}}{\sigma^2_{\mathsf{M}}} + \frac{\sigma^2_{\varepsilon}}{\sigma^2_{\mathsf{M}}}\right) \tag{67}$$

$$= \frac{\sigma^2_{\mathsf{U}}}{\sigma^2_{\mathsf{M}}} \frac{\sigma^2_{\mathsf{M}} + \mu^2_{\mathsf{M}}}{\sigma^2_{\mathsf{M}}} \left(\beta_{\tau,n} - \beta_{\mathsf{U},n}\right)^2 \sigma^2_{\tau} \tag{68}$$

$$+ \frac{\sigma^2_{\tau} + \mu^2_{\mathsf{M}}}{\mu^2_{\mathsf{M}}} \frac{\sigma^2_{\mathsf{M}}}{\sigma^2_{\mathsf{U}}} \cdot T\,\text{VAR}\left[\mu_{\mathsf{M}} \frac{\sigma^2_{\mathsf{U}}}{\sigma^2_{\mathsf{M}}} \left(\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}\right)\right] \tag{69}$$

$$\frac{\text{VAR}\left[\hat{\alpha}_n\right]}{\text{VAR}\left[\bar{r}_{\tau} \frac{\hat{\sigma}^2_{\mathsf{U}}}{\hat{\sigma}^2_{\mathsf{M}}} \left(\hat{\beta}_{\tau,n} - \hat{\beta}_{\mathsf{U},n}\right)\right]} \to \left(\left(\beta_{\tau,n} - \beta_{\mathsf{U},n}\right)^2 \frac{\sigma^2_{\tau}}{\sigma^2_{\varepsilon}} + \frac{\sigma^2_{\mathsf{M}}}{\sigma^2_{\mathsf{U}}}\right) \frac{\sigma^2_{\tau} + \mu^2_{\mathsf{M}}}{\mu^2_{\mathsf{M}}} \tag{70}$$

$$= \left(\left(\frac{\alpha^2_n}{\sigma^2_{\varepsilon}} \frac{\sigma^2_{\mathsf{M}}}{\mu^2_{\mathsf{M}}} + 1\right) \cdot \frac{\sigma^2_{\tau}}{\sigma^2_{\mathsf{U}}} + 1\right) \cdot \left(\frac{\sigma^2_{\tau}}{\mu^2_{\mathsf{M}}} + 1\right) > 1 \tag{71}$$

Asymptotically, the alpha has higher variance than the beta-based measure of mispricing. In addition, the asymptotic variance ratio is greater for stocks that have larger mispricing errors, and increases with α^2_n . Intuitively, stocks with equal parts smart and dumb beta consists of market risk r_m and purely idiosyncratic shocks ε. No additional information is gained from having the smart and dumb factors.

A.2 Sharper Alpha with Measurement Error

Take Y the vector of T observations of excess returns and $\tilde{X} = [\tilde{r}_\tau | \tilde{r}_{\mathsf{U}}]$ where

$$\tilde{r}_\tau = r_\tau + \xi \tag{72}$$

$$\tilde{r}_{\mathsf{U}} = r_{\mathsf{M}} - \tilde{r}_\tau \tag{73}$$

$$= r_{\mathsf{U}} - \xi \tag{74}$$

and ξ is the measurement error in the estimated tangency factor $\tilde{r}_\tau$. We can write the returns as

$$Y = \tilde{X}\tilde{\gamma} + u \tag{75}$$

$$\tilde{\gamma} = \begin{bmatrix} \tilde{\beta}_{\tau,n} \\ \tilde{\beta}_{\mathsf{U},n} \end{bmatrix} \tag{76}$$

.

The sample estimates for the coefficient is

$$\hat{\gamma} = \left(\tilde{X}^\intercal \tilde{X}\right)^{-1} \tilde{X}^\intercal Y.$$

Consider a case where the estimated tangency factor earns the same risk premium as the market, and is uncorrelated with the estimated unpriced factor r_{U}. With this additional restriction on estimation noise, we have

$$0 = \sigma_{\tau,\xi} \tag{77}$$

$$0 = \sigma_{\tau,\mathsf{U}} - \sigma_{\tau,\xi} + \sigma_{\mathsf{U},\xi} - \sigma_{\xi}^{2} \tag{78}$$

$$\sigma_{\mathsf{U},\xi} = \sigma_{\xi}^{2} \tag{79}$$

$$T^{-1}\tilde{r}_{\tau}^{\intercal}\tilde{r}_{\tau} \to \mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2} \tag{80}$$

$$T^{-1}\tilde{r}_{\mathsf{U}}^{\intercal}\tilde{r}_{\mathsf{U}} \to \mathrm{VAR}\left[\tilde{r}_{\mathsf{U}}\right] = \sigma_{\mathsf{U}}^{2} - \sigma_{\xi}^{2} \tag{81}$$

$$T^{-1}\tilde{r}_{\tau}^{\intercal}\tilde{r}_{\mathsf{U}} \to 0 \tag{82}$$

.

The estimated difference in betas converges to

$$\begin{bmatrix} 1 & -1 \end{bmatrix}\hat{\gamma} \to \left(1 - \frac{\sigma_{\xi}^{2}}{\mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2}}\right) \cdot \left(\beta_{\tau,n} - \beta_{\mathsf{U},n}\right) + \frac{\sigma_{\xi}^{2}}{\mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2}} \cdot \frac{\sigma_{\xi,\varepsilon_n}}{\sigma_{\xi}^{2}} \cdot \frac{\sigma_{\mathsf{M}}^{2}}{\mathrm{VAR}\left[\tilde{r}_{\mathsf{U}}\right]} \tag{83}$$

$$\hat{\alpha}_{n}^{*} \to \left(1 - \frac{\sigma_{\xi}^{2}}{\mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2}}\right)\left(1 - \frac{\sigma_{\xi}^{2}}{\sigma_{\mathsf{U}}^{2}}\right) \cdot \alpha_{n} + \frac{\sigma_{\xi}^{2}}{\mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2}} \cdot \frac{\sigma_{\xi,\varepsilon_n}}{\sigma_{\xi}^{2}} \cdot \mu_{\mathsf{M}} \tag{84}$$

where σ_{ξ}^{2} is the amount of measurement error in the estimated $\tilde{r}_{\tau} = r_{\tau} + \xi$ and $\sigma_{\xi,\varepsilon_n}$ is the covariance between the measurement error ξ and the stock residuals ε.

If the stock residuals are uncorrelated with the measurement error, we get an attenuation bias that shrinks all sharper alphas towards zero. We then get

$$\hat{\alpha}_{n}^{*} \xrightarrow[T\to\infty]{} \left(1 - \pi_{1}\right)\left(1 - \pi_{2}\right) \cdot \alpha_{n} + \pi_{1} \cdot \beta_{\xi,n} \cdot \mu_{\mathsf{M}} \tag{85}$$

where

$$\pi_{1} = \frac{\mathrm{E}\left[\xi^{2}\right]}{\mathrm{E}\left[\hat{r}_{\tau}^{2}\right]} = \frac{\sigma_{\xi}^{2}}{\mu_{\mathsf{M}}^{2} + \sigma_{\tau}^{2} + \sigma_{\xi}^{2}} \in [0,1] \tag{86}$$

is the amount of noise in the estimated tangency factor,

$$\pi_2 = \frac{\mathrm{E}\left[\xi^2\right]}{\mathrm{E}\left[r_{\mathsf{U}}^2\right]} = \frac{\sigma_\xi^2}{\sigma_{\mathsf{U}}^2} \in [0,1]$$

is the amount of noise relative to the unpriced factor, and $\beta_{\xi,n} = \frac{\mathrm{cov}[\xi,\varepsilon_n]}{\sigma_\xi^2}$ is each stock's direct loading on the measurement error.

The attenuation bias, as a function of measurement error σ_ξ, is plotted in Figure A.1 using the sample moments for μ_{M}^2, σ_τ^2, and σ_{U}^2, calculated with daily returns from 1967 to 2019.

A.3 GRS Joint test

To jointly test that N assets all have zero alpha, we can stack their returns into an N vector $\boldsymbol{r_t}$. With the beta decomposition, we have

$$\boldsymbol{r}_t = \boldsymbol{\beta}_\tau \cdot r_\tau + \boldsymbol{\beta}_{\mathsf{U}} \cdot r_{\mathsf{U}} + \boldsymbol{u}_t \tag{87}$$

Setting the GMM conditions to

$$\boldsymbol{b} = \begin{bmatrix} \boldsymbol{\beta}_\tau \\ \boldsymbol{\beta}_{\mathsf{U}} \end{bmatrix} \tag{88}$$

$$\boldsymbol{g}_T\left(\boldsymbol{b}\right) = \mathrm{E}\begin{bmatrix} \boldsymbol{r} - \boldsymbol{\beta}_\tau \cdot r_\tau - \boldsymbol{\beta}_{\mathsf{U}} \cdot r_{\mathsf{U}} \\ \left(\boldsymbol{r} - \boldsymbol{\beta}_\tau \cdot r_\tau - \boldsymbol{\beta}_{\mathsf{U}} \cdot r_{\mathsf{U}}\right) \cdot f_t \end{bmatrix} \tag{89}$$

$$= \begin{bmatrix} \mathrm{E}\left[\boldsymbol{u}_t\right] \\ \mathrm{E}\left[\boldsymbol{u}_t \cdot f_t\right] \end{bmatrix} = \mathbf{0} \tag{90}$$

$$\boldsymbol{d} = \frac{\partial}{\partial \boldsymbol{b}^\intercal} \boldsymbol{g}_T\left(\boldsymbol{b}\right) \tag{91}$$

$$= -\begin{bmatrix} \mathrm{E}\left[r_{\tau,t}^2\right] & \mathrm{E}\left[r_{\mathsf{U},t} r_{\tau,t}\right] \\ \mathrm{E}\left[r_{\mathsf{U},t} r_{\tau,t}\right] & \mathrm{E}\left[r_{\mathsf{U},t}^2\right] \end{bmatrix} \otimes I_N \tag{92}$$

and assuming no serial correlation, the weighing matrix is

$$S = \mathrm{E}\left[\begin{array}{cc} r_{\tau,t}u_t u_t^{\intercal} r_{\tau,t} & r_{\mathsf{U},t}u_t u_t^{\intercal} r_{\tau,t} \\ r_{\tau,t}u_t u_t^{\intercal} r_{\mathsf{U},t} & r_{\mathsf{U},t}u_t u_t^{\intercal} r_{\mathsf{U},t} \end{array}\right]$$

which further simplifies assuming independence between the error and the factor

$$S = \left[\begin{array}{cc} \mathrm{E}\left[r_{\tau,t}^2\right] & \mathrm{E}\left[r_{\mathsf{U},t} r_{\tau,t}\right] \\ \mathrm{E}\left[r_{\mathsf{U},t} r_{\tau,t}\right] & \mathrm{E}\left[r_{\mathsf{U},t}^2\right] \end{array}\right] \otimes \mathrm{E}\left[u_t u_t^{\intercal}\right]. \tag{93}$$

The variance of the GMM estimator is

$$\mathrm{VAR}\left[b\right] = \frac{1}{T} d^{-1} S d^{-1} \tag{94}$$

and simplifies to

$$\mathrm{VAR}\left[b\right] = \frac{1}{T}\frac{1}{\left(\mu_\tau^2 + \sigma_\tau^2\right)\sigma_{\mathsf{U}}^2}\left[\begin{array}{cc} \sigma_{\mathsf{U}}^2 & 0 \\ 0 & \mu_\tau^2 + \sigma_\tau^2 \end{array}\right] \otimes \mathrm{E}\left[u_t u_t^{\intercal}\right]. \tag{95}$$

The sharper alpha has a variance of

$$\begin{aligned}
\mathrm{VAR}\left[\begin{array}{cc} \boldsymbol{\beta}_\tau - & \boldsymbol{\beta}_{\mathsf{U}} \end{array}\right] &= \frac{1}{T}\frac{\mu_\tau^2 + \sigma_\tau^2 + \sigma_{\mathsf{U}}^2}{\left(\mu_\tau^2 + \sigma_\tau^2\right)\sigma_{\mathsf{U}}^2}\mathrm{E}\left[u_t u_t^{\intercal}\right] && (96)\\
&= \frac{1}{T}\frac{\mu_{\mathsf{M}}^2 + \sigma_{\mathsf{M}}^2}{\left(\mu_\tau^2 + \sigma_\tau^2\right)\sigma_{\mathsf{U}}^2}\mathrm{E}\left[u_t u_t^{\intercal}\right] && (97)\\
&= \frac{1}{T}\frac{1}{\mu_\tau^2 + \sigma_\tau^2} \cdot \frac{\sigma_{\mathsf{M}}^2}{\sigma_{\mathsf{U}}^2} \cdot \left(1 + \frac{\mu_{\mathsf{M}}^2}{\sigma_{\mathsf{M}}^2}\right)\mathrm{E}\left[u_t u_t^{\intercal}\right] && (98)
\end{aligned}$$

The joint test statistic is then

$$
\begin{aligned}
J_{\mathrm{s}} &= \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \ \ \hat{\boldsymbol{\beta}}_{\mathsf{U}} \ \right]^{\mathsf{T}} \mathrm{VAR}\left[\ \hat{\boldsymbol{\beta}}_{\tau} - \ \ \hat{\boldsymbol{\beta}}_{\mathsf{U}} \ \right]^{-1} \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \ \ \hat{\boldsymbol{\beta}}_{\mathsf{U}} \ \right] && (99) \\
&= T \cdot \left(1 + \frac{\bar{r}_{\mathsf{M}}^2}{\hat{\sigma}_{\mathsf{M}}^2}\right)^{-1} \cdot \left(1 + \frac{\bar{r}_{\tau}^2}{\hat{\sigma}_{\tau}^2}\right) \cdot \frac{\hat{\sigma}_{\tau}^2 \hat{\sigma}_{\mathsf{U}}^2}{\hat{\sigma}_{\mathsf{M}}^2} \cdot \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \ \ \hat{\boldsymbol{\beta}}_{\mathsf{U}} \ \right]^{\mathsf{T}} \mathrm{VAR}\left[\hat{u}\right]^{-1} \left[\ \hat{\boldsymbol{\beta}}_{\tau} - \ \ \hat{\boldsymbol{\beta}}_{\mathsf{U}} \ \right] && (100) \\
J_{\mathrm{s}} &\xrightarrow{a} \chi_N^2 && (101)
\end{aligned}
$$

with the finite sample analogue under normality being

$$
\begin{aligned}
F_{\mathrm{s}} &= \frac{T - K - N}{N} \frac{J_{\mathrm{s}}}{T - K - 1} \\
F_{\mathrm{s}} &\overset{d}{\sim} F_{N, T-N-K}.
\end{aligned}
$$

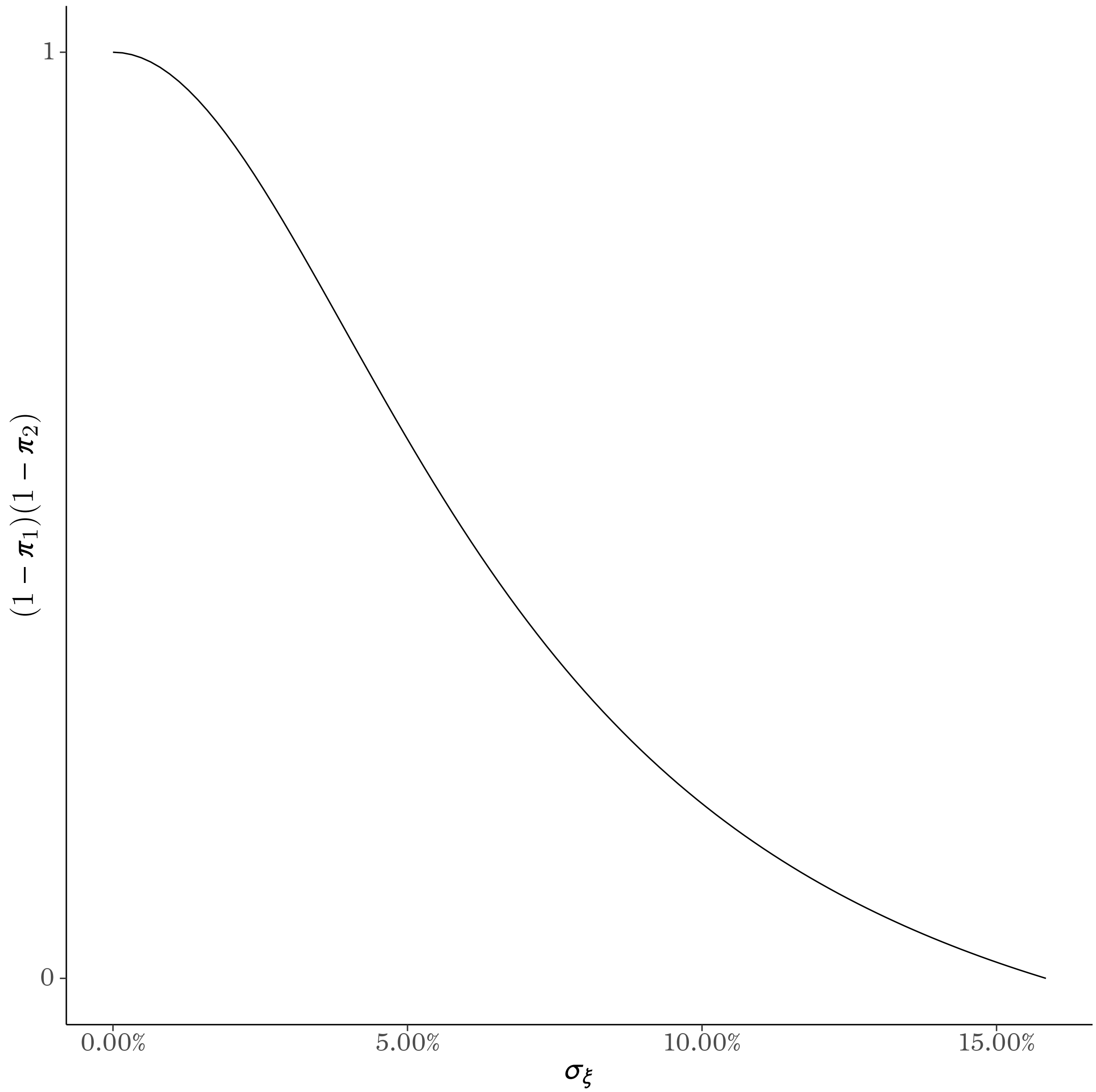

Figure A.1: **Attenuation from measurement error.** This figure illustrates the attenuation bias from Proposition (7). As measurement error increases, sharper alphas shrink towards zero.

B Anomaly-Sorted Portfolio Alphas

The tables in this section report alphas and their t-statistics for 10 portfolios sorted beta (beta_1), boot-to-market (bm), earnings-to-price (ep), investment-to-asset (ia), size (me), momentum (r11_1) and return on equity (row_1), estimated from daily returns between 1967 to 2018.

Daily portfolio returns were retrieved from the Hou-Xue-Zhang q-factors data library at global-q.org.

	capm α %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	2.30	2.38	1.52	2.31	1.18	0.83	0.31	−0.81	−2.30	−4.66
	(2.25)	(2.36)	(1.52)	(2.43)	(1.31)	(0.93)	(0.34)	(−0.80)	(−1.99)	(−2.76)
bm	−1.51	0.17	1.48	0.64	1.21	1.68	2.45	2.42	2.83	4.29
	(−1.82)	(0.24)	(1.90)	(0.77)	(1.31)	(1.82)	(2.52)	(2.44)	(2.63)	(3.12)
ep	−1.90	−0.64	0.59	1.95	0.30	1.80	3.51	3.18	3.25	3.40
	(−1.79)	(−0.78)	(0.68)	(2.32)	(0.36)	(1.94)	(3.95)	(3.01)	(2.91)	(2.72)
ia	1.94	2.75	2.74	1.53	1.30	1.14	1.24	0.32	0.03	−3.69
	(1.84)	(2.81)	(3.13)	(1.83)	(1.65)	(1.39)	(1.50)	(0.40)	(0.03)	(−3.39)
me	0.66	0.58	0.87	1.16	1.15	0.60	1.38	0.96	0.78	−0.16
	(0.48)	(0.41)	(0.68)	(0.98)	(1.08)	(0.66)	(1.65)	(1.33)	(1.33)	(−0.31)
r11_1	−10.02	−2.10	−0.80	0.11	−0.02	0.47	0.52	1.94	2.62	5.49
	(−5.22)	(−1.52)	(−0.71)	(0.11)	(−0.03)	(0.56)	(0.61)	(2.20)	(2.59)	(3.65)
roe_1	−7.60	−2.58	−1.52	−0.32	1.15	−0.36	1.14	0.80	0.60	2.64
	(−5.06)	(−2.33)	(−1.51)	(−0.36)	(1.39)	(−0.43)	(1.47)	(1.05)	(0.78)	(2.96)

	capm α^* %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	1.49	0.63	0.60	0.97	0.66	0.29	0.36	−0.24	−1.26	−3.41
	(12.05)	(5.12)	(4.91)	(8.47)	(6.07)	(2.74)	(3.24)	(−1.96)	(−9.00)	(−16.81)
bm	−2.21	−0.89	−0.03	0.14	0.31	0.66	0.97	1.61	2.09	2.59
	(−22.37)	(−10.56)	(−0.36)	(1.42)	(2.75)	(5.86)	(8.24)	(13.46)	(16.16)	(15.65)
ep	−2.65	−0.36	−0.38	0.04	0.36	0.68	1.43	1.15	1.33	1.61
	(−20.90)	(−3.58)	(−3.60)	(0.43)	(3.49)	(6.03)	(13.39)	(8.97)	(9.82)	(10.64)
ia	1.66	1.34	1.23	0.84	1.05	0.56	0.30	−0.47	−2.00	−3.25
	(13.00)	(11.38)	(11.62)	(8.31)	(11.08)	(5.69)	(2.99)	(−4.83)	(−16.60)	(−25.19)
me	0.30	1.53	1.50	1.28	1.25	0.55	−0.10	−0.09	0.07	−1.09
	(1.78)	(9.00)	(9.64)	(8.95)	(9.64)	(5.01)	(−0.98)	(−1.02)	(0.95)	(−17.19)
r11_1	−7.13	−2.20	−0.64	−0.77	−0.64	−0.13	0.22	0.74	1.41	2.43
	(−31.73)	(−13.25)	(−4.70)	(−6.67)	(−5.80)	(−1.31)	(2.11)	(6.91)	(11.54)	(13.40)
roe_1	−5.27	−2.12	−1.55	−0.68	−0.11	−0.32	0.02	0.05	0.09	0.39
	(−29.89)	(−15.92)	(−12.86)	(−6.25)	(−1.09)	(−3.05)	(0.22)	(0.56)	(0.91)	(3.63)

	ff3 α %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	1.59	2.23	1.42	2.04	0.89	0.49	−0.15	−1.18	−2.32	−4.02
	(1.60)	(2.30)	(1.46)	(2.17)	(0.99)	(0.56)	(−0.16)	(−1.17)	(−2.16)	(−2.68)
bm	1.04	1.30	1.90	0.43	0.49	0.61	1.01	0.42	0.37	1.10
	(1.75)	(2.02)	(2.46)	(0.52)	(0.54)	(0.69)	(1.10)	(0.48)	(0.40)	(0.95)
ep	0.26	0.49	1.41	2.37	0.41	1.18	3.03	1.94	1.58	1.12
	(0.28)	(0.63)	(1.65)	(2.84)	(0.49)	(1.30)	(3.46)	(1.91)	(1.51)	(0.99)
ia	1.18	1.65	2.27	1.31	0.98	1.39	1.40	1.40	1.89	−1.77
	(1.16)	(1.74)	(2.66)	(1.57)	(1.26)	(1.73)	(1.74)	(1.84)	(2.10)	(−1.89)
me	−1.10	−1.46	−0.71	−0.11	0.11	−0.06	0.94	0.70	0.74	1.10
	(−1.41)	(−2.66)	(−1.52)	(−0.23)	(0.22)	(−0.11)	(1.45)	(1.12)	(1.30)	(3.32)
r11_1	−11.49	−3.16	−1.73	−0.46	−0.45	0.27	0.34	2.10	3.14	7.32
	(−6.28)	(−2.33)	(−1.57)	(−0.49)	(−0.50)	(0.33)	(0.40)	(2.38)	(3.13)	(5.41)
roe_1	−7.58	−3.62	−2.31	−0.79	0.82	−0.22	1.35	1.65	2.05	4.28
	(−5.70)	(−3.48)	(−2.35)	(−0.89)	(0.99)	(−0.26)	(1.74)	(2.25)	(2.97)	(5.29)

	ff3 α^* %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	0.79	0.57	0.59	0.73	0.37	−0.08	−0.17	−0.71	−1.46	−3.04
	(6.88)	(5.10)	(5.25)	(6.78)	(3.54)	(−0.76)	(−1.61)	(−6.16)	(−11.91)	(−17.83)
bm	0.57	0.35	0.43	−0.09	−0.49	−0.53	−0.60	−0.56	−0.59	−0.96
	(8.35)	(4.67)	(4.83)	(−0.99)	(−4.61)	(−5.20)	(−5.73)	(−5.54)	(−5.53)	(−7.17)
ep	−0.37	0.87	0.53	0.53	0.53	0.04	0.96	−0.18	−0.47	−0.91
	(−3.51)	(9.76)	(5.37)	(5.57)	(5.49)	(0.43)	(9.57)	(−1.51)	(−3.91)	(−6.96)
ia	0.74	0.13	0.78	0.64	0.76	0.90	0.55	0.73	−0.03	−1.29
	(6.34)	(1.22)	(7.95)	(6.66)	(8.56)	(9.78)	(5.88)	(8.28)	(−0.33)	(−12.06)
me	−2.06	−1.16	−0.67	−0.50	−0.24	−0.43	−0.79	−0.51	−0.03	0.41
	(−23.36)	(−18.61)	(−12.48)	(−9.43)	(−4.14)	(−6.23)	(−10.72)	(−7.04)	(−0.52)	(10.70)
r11_1	−9.02	−3.44	−1.66	−1.37	−1.07	−0.33	0.06	0.94	1.96	4.28
	(−46.19)	(−22.51)	(−13.22)	(−12.75)	(−10.36)	(−3.41)	(0.61)	(9.33)	(17.20)	(28.33)
roe_1	−5.57	−3.41	−2.49	−1.23	−0.49	−0.16	0.25	1.02	1.72	2.23
	(−38.41)	(−29.34)	(−22.52)	(−12.08)	(−5.15)	(−1.63)	(2.78)	(12.17)	(21.93)	(24.48)

	q-factor α %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	−0.52	−0.73	−1.42	−0.78	−0.83	−0.34	−0.20	−0.11	0.11	1.75
	(−0.54)	(−0.80)	(−1.54)	(−0.87)	(−0.94)	(−0.38)	(−0.22)	(−0.11)	(0.10)	(1.22)
bm	−0.15	−0.15	0.19	−0.28	−0.27	1.08	1.79	1.00	1.76	4.11
	(−0.21)	(−0.22)	(0.25)	(−0.34)	(−0.30)	(1.23)	(1.95)	(1.09)	(1.80)	(3.45)
ep	1.64	−1.70	−1.15	0.44	−1.25	−0.03	1.27	1.29	1.93	2.72
	(1.71)	(−2.12)	(−1.36)	(0.54)	(−1.53)	(−0.03)	(1.48)	(1.26)	(1.76)	(2.22)
ia	−0.56	0.29	−0.64	0.01	−0.28	−0.55	−0.50	1.37	3.33	0.44
	(−0.60)	(0.36)	(−0.89)	(0.01)	(−0.37)	(−0.70)	(−0.64)	(1.82)	(3.98)	(0.49)
me	0.05	−0.73	−0.11	0.67	0.98	0.87	2.08	1.77	1.21	0.21
	(0.06)	(−1.19)	(−0.21)	(1.28)	(1.79)	(1.37)	(3.16)	(2.81)	(2.10)	(0.64)
r11_1	−1.65	2.06	1.02	0.48	−0.55	−0.57	−1.89	−0.77	−0.42	4.45
	(−1.02)	(1.59)	(0.94)	(0.51)	(−0.61)	(−0.69)	(−2.31)	(−0.92)	(−0.43)	(3.19)
roe_1	0.76	2.34	2.89	1.16	1.56	0.36	1.05	−0.51	−0.45	−0.10
	(0.71)	(2.77)	(3.62)	(1.35)	(1.93)	(0.42)	(1.35)	(−0.71)	(−0.67)	(−0.14)

	q-factor α^* %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	−0.41	−1.20	−1.08	−0.98	−0.65	−0.50	−0.11	0.05	0.05	0.44
	(−4.86)	(−15.16)	(−13.48)	(−12.60)	(−8.41)	(−6.51)	(−1.42)	(0.59)	(0.58)	(3.45)
bm	−0.69	−0.83	−0.78	−0.52	−0.81	−0.03	0.19	0.25	0.80	1.51
	(−11.34)	(−13.95)	(−11.63)	(−7.16)	(−10.25)	(−0.45)	(2.33)	(3.16)	(9.40)	(14.53)
ep	0.05	−0.84	−1.28	−0.80	−0.53	−0.56	−0.12	−0.26	0.20	0.67
	(0.58)	(−12.06)	(−17.35)	(−11.07)	(−7.34)	(−7.11)	(−1.57)	(−2.95)	(2.06)	(6.28)
ia	−0.54	−0.65	−1.04	−0.22	−0.004	−0.41	−0.64	0.49	0.56	−0.20
	(−6.63)	(−8.97)	(−16.75)	(−3.14)	(−0.06)	(−5.91)	(−9.31)	(7.50)	(7.62)	(−2.55)
me	−1.40	−0.90	−0.58	−0.33	−0.01	−0.06	−0.22	0.04	0.18	−0.26
	(−19.90)	(−16.92)	(−12.39)	(−7.18)	(−0.11)	(−1.01)	(−3.81)	(0.80)	(3.62)	(−8.77)
r11_1	−1.78	0.36	0.46	−0.46	−0.85	−0.74	−1.26	−0.96	−0.62	1.39
	(−12.59)	(3.22)	(4.81)	(−5.55)	(−10.79)	(−10.26)	(−17.77)	(−13.09)	(−7.29)	(11.38)
roe_1	−0.16	0.70	1.07	0.16	0.04	0.20	−0.01	−0.54	−0.24	−0.98
	(−1.71)	(9.42)	(15.41)	(2.10)	(0.55)	(2.66)	(−0.08)	(−8.58)	(−4.03)	(−15.10)

	ff5 α %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	−0.74	−1.13	−2.01	−1.30	−1.60	−1.28	−1.43	−1.87	−1.16	0.47
	(−0.79)	(−1.32)	(−2.35)	(−1.57)	(−1.93)	(−1.51)	(−1.59)	(−1.88)	(−1.09)	(0.34)
bm	1.26	0.41	0.39	−1.02	−1.24	−0.11	0.07	−1.03	−0.70	0.36
	(2.17)	(0.65)	(0.52)	(−1.26)	(−1.41)	(−0.13)	(0.08)	(−1.20)	(−0.78)	(0.32)
ep	2.43	−0.79	−0.68	0.51	−1.44	−0.84	0.83	−0.50	−0.47	−0.17
	(2.79)	(−1.04)	(−0.84)	(0.64)	(−1.80)	(−0.97)	(1.00)	(−0.52)	(−0.46)	(−0.15)
ia	−0.53	−0.54	−0.82	−0.26	−0.61	−0.54	−0.72	1.44	3.92	0.57
	(−0.61)	(−0.67)	(−1.22)	(−0.33)	(−0.82)	(−0.70)	(−0.95)	(1.93)	(4.80)	(0.67)
me	−0.78	−1.53	−0.62	0.20	0.74	0.48	1.77	1.43	1.01	0.46
	(−1.00)	(−2.83)	(−1.33)	(0.43)	(1.51)	(0.80)	(2.80)	(2.35)	(1.79)	(1.44)
r11_1	−8.45	−2.45	−2.14	−1.75	−2.30	−1.17	−2.05	−0.08	1.13	8.28
	(−4.73)	(−1.82)	(−1.96)	(−1.89)	(−2.65)	(−1.45)	(−2.60)	(−0.10)	(1.17)	(6.14)
roe_1	−2.82	−1.94	−1.04	−0.14	0.71	−0.14	0.70	0.54	1.10	1.97
	(−2.48)	(−1.94)	(−1.10)	(−0.16)	(0.89)	(−0.17)	(0.90)	(0.75)	(1.64)	(2.68)

	ff5 α^* %									
	rank									
	1	2	3	4	5	6	7	8	9	10
beta_1	−0.50	−1.28	−1.35	−1.17	−1.07	−1.09	−0.89	−1.12	−0.79	−0.42
	(−5.12)	(−14.50)	(−15.30)	(−13.74)	(−12.49)	(−12.38)	(−9.61)	(−10.90)	(−7.14)	(−2.94)
bm	0.60	−0.15	−0.40	−0.87	−1.37	−0.85	−1.03	−1.30	−1.10	−1.26
	(10.04)	(−2.29)	(−5.23)	(−10.39)	(−15.13)	(−9.40)	(−11.14)	(−14.70)	(−11.77)	(−10.60)
ep	0.82	0.13	−0.65	−0.50	−0.51	−1.07	−0.29	−1.53	−1.63	−1.70
	(9.11)	(1.59)	(−7.72)	(−6.01)	(−6.14)	(−11.86)	(−3.34)	(−15.42)	(−15.62)	(−14.92)
ia	−0.04	−0.92	−0.83	−0.17	−0.12	−0.19	−0.67	0.65	0.97	−0.04
	(−0.44)	(−10.98)	(−11.90)	(−2.07)	(−1.51)	(−2.41)	(−8.59)	(8.35)	(11.49)	(−0.44)
me	−1.74	−1.10	−0.55	−0.28	0.15	−0.08	−0.26	−0.03	0.16	0.04
	(−22.09)	(−19.95)	(−11.56)	(−5.96)	(2.89)	(−1.32)	(−3.99)	(−0.48)	(2.69)	(1.18)
r11_1	−7.00	−2.96	−1.86	−2.06	−2.06	−1.11	−1.24	−0.28	0.79	4.66
	(−40.01)	(−21.47)	(−16.58)	(−21.75)	(−23.33)	(−13.26)	(−15.30)	(−3.27)	(7.89)	(34.69)
roe_1	−2.57	−2.23	−1.60	−0.72	−0.44	−0.06	−0.10	0.35	1.06	0.79
	(−22.15)	(−21.97)	(−16.41)	(−8.10)	(−5.34)	(−0.68)	(−1.27)	(4.69)	(15.38)	(10.32)